DEDICATION

This work is dedicated to a number of people not because of its importance but because without them it would have been meaningless to pursue.

First to the homeless, not because it will be useful to them but because I share some of their sentiments. To my friend and colleague at U.B.C. who had shown me perseverance by his dedication and devotion to research despite living in poverty and his twelve year old son. To my friends in Vancouver and here who gave me their companionship, their warmth, in particular, their encouragement and their care. To my parents and to my grandmother who taught me the values of human endeavours and toils. And last but not least to my childhood teachers, the sandal maker, the street hawker and vegetable grower.

Tze-Beng Ng
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at
Coventry.

ABSTRACT

This thesis is an effort to verify a conjecture of E. Thomas, namely the following :

Given a smooth, orientable, connected and closed manifold satisfying the following conditions:

(A) M is an (odd) n-dimensional manifold which is (k-2)-connected where k is a positive integer satisfying

$$k \leq n/2 , \rho(n)$$

where $\rho(n) = 2^b+8a-1$ if $(n+1) = 2^{4a+b}.$(odd integer), $0 \leq b \leq 3$;

and (B) $w_{k-1}(M) = w_k(M) = 0$ and either

$w_{n-k+1}(M) = 0$ or $\delta w_{n-k}(M) = 0$ where δ is the Bockstein coboundary operator associated with the sequence $0 \longrightarrow Z \to Z \to Z_2 \to 0$, depending on when n-k is odd or even. Then M admits k linearly independent tangent vector fields without singularities.

The case k=7 is studied in detail and the conjecture is shown to be true in this case. The method follows Mahowald and E. Thomas' approach to lifting a map. Roughly speaking, we consider this as a lifting problem: namely, finding a lifting of the classifying map of the tangent bundle (considered as a spin bundle), $\tau : M \longrightarrow BSpin_n$ to $BSpin_{n-7}$. The proof works as follows: first we consider $\tau : M \longrightarrow BSpin_n$ as a map from M to $BSpin_n[8]$ where $BSpin_n[8]$ is the 7-connective covering over $BSpin_n$; and we seek lifting of this map to $BSpin_{n-7}[8]$. An n-MPT is

constructed for the fibration $BSpin_{n-7}[8] \longrightarrow BSpin_n[8]$. An admissible class theorem is proved, which enables the identification of the final obstruction to lifting as a tertiary cohomology operation Θ^* of degree n, valid only on integral classes, applied to the Thom class U_τ of the tangent bundle of M with zero indeterminacy. Following Milnor, we consider the Thom class as $\underline{U} \in H^n(M \times M)$. We consider another operation $\tilde{\Omega}^*$ which contains Θ^*; and we show that $\tilde{\Omega}^*(U_\tau) = \Theta^*(U_\tau)$ modulo zero indeterminacy. Next we show that there is a decomposition $\underline{U} = A + t A$ where $A \in H^n(M \times M)$ and $t : H^*(M \times M) \rightarrow H^*(M \times M)$ is induced by the map that interchanges the factors. We observe that $\approx{\Omega}$ is defined on A. Hence $\approx{\Omega}(\underline{U}) = 0$ which implies that $\tilde{\Omega}^*(U_\tau) = 0$. Thus the existence of a lifting of τ to $BSpin_{n-7}[8]$ follows.

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CHAPTER 1. OBSTRUCTION THEORY AND THE k-INVARIANTS

In this chapter, we shall develop the obstruction theory for liftings in so far as is needed. We shall assume familiarity with the language of fibre bundles and the theory of classifying spaces.

Let k be a positive integer. Consider the k-frame bundle associated with the tangent bundle τ_M of a smooth manifold M of dimension n. Call it $E_k \longrightarrow M$. According to N. Steenrod [26], τ_M admits a k-field iff E_k admits a section. So our aim is to determine the obstruction to sectioning E_k and deduce some sufficient condition for E_k to admit a section.

An n-plane bundle is classified by a map into BO_n, the classifying space for the n-dimensional orthogonal group O(n). The associated k-frame bundle to the universal n-plane bundle over BO_n is well known to be the same as the inclusion $BO_{n-k} \longrightarrow BO_n$. Thus by the pullback property, E_k has a section iff the classifying map $\tau_M : M \longrightarrow BO_n$ of the tangent bundle lifts to BO_{n-k}. We will need therefore a Postnikov resolution for the fibration, $BO_{n-k} \longrightarrow BO_n$ [17]. Since the manifold we consider will always be orientable, we actually constructed a modified Postnikov tower for the fibration, $BSO_{n-k} \longrightarrow BSO_n$, where BSO_n is the classifying space for the special orthogonal group, SO(n).

The aim of this work is to prove E. Thomas' conjecture (see §1 below) for the case when k is seven and the dimension of the manifold is congruent to seven modulo eight and is greater or equal to fifteen. The method follows Thomas-Mahowald approach to lifting. It is shown that the obstruction to the existence of a 7-field is given by a tertiary cohomology operation of degree n $\tilde{\Omega}^*$ on the Thom class U_τ of the tangent bundle of the manifold in question. There is a map $g: M\times M \longrightarrow T_\tau$, where T_τ is the Thom complex of the tangent bundle, which pinches the complement of a tubular neighbourhood of the diagonal in $M\times M$ to a point. We observe that $\tilde{\Omega}^*(U_\tau) = 0$ iff $\Omega(g^*(U_\tau)) = 0$ where Ω is the corresponding tertiary cohomology operation associated with similar relation which determined $\tilde{\Omega}^*$ but which hold on all mod 2 classes. The proof consists of showing that $\Omega(g^*(U_\tau))$ is zero modulo zero indeterminacy.

In Chapter 2, the k-invariants for the second stage of a modified Postnikov tower is related to some secondary cohomology operations on the Thom complex. The computation of some of the secondary cohomology operations is continued in Chapter 3 and a Thomas type admissible class theorem is proved in Chapter 4. The final obstruction is identified as a tertiary (stable) cohomology operation $\textcircled{H}^*$ (valid only on integral classes) applied to U_τ. A modification of this operation is then made in Chapter 5 and it is shown that the new operation also detects the final obstruction and that it is zero on U . Hence the existence of 7 linearly independent tangent vector fields follows. We close the chapter with an observation that the dual operation of (H) is always zero on $U_{BSpin_N[8]}$ the Thom class of the universal N-plane bundle over $BSpin_N[8]$ when $N > 15$.

Throughout this dissertation cohomology will always be ordinary cohomology with coefficients in mod 2 integers unless otherwise specified. The manifold M will always be connected, closed, orientable and smooth. n will always denote the dimension of M. The following conventional notation is used. ι_k will always denote the fundamental class of an Eilenberg-MacLane complex of type (π, q) for any Abelian group π and any positive integer q , or a component of a generalized Eilenberg-MacLane complex (i.e. a product of Eilenberg-MacLane complexes of type (π, k), where π is a cyclic Abelian group of prime order p. $K(Z_2,k)$ will sometimes be denoted by K_k (Eilenberg-MacLane complex of type (Z_2,k)) and $K(Z,k)$ by K_k^*. The integers are denoted by Z as is assumed above and mod p integers by Z_p for p a prime. All maps are continuous.

§ 1.1. The Conjecture of E. Thomas

1.1.1. The Conjecture.

Let M be a manifold of dimension n odd > 0. Suppose M is closed, connected, orientable and smooth. Let k be a positive integer and $0 < k < n$. We consider the following conditions:

CONDITION (A). M is $(k-2)$-connected, i.e. $\pi_i(M)=0$ for $0 \leq i \leq k-2$.

CONDITION (B). The $(k-1)^{th}$ and k^{th} mod 2 Stiefel-Whitney classes of the tangent bundle of M satisfy

$$w_{k-1}(M)=w_k(M)=0$$

and either the $(n-k+1)^{th}$ mod 2 Stiefel Whitney class $w_{n-k+1}(M)=0$ or the $(n-k+1)^{th}$ integral Stiefel Whitney class $W_{n-k+1}(M)=\delta w_{n-k}(M)=0$ depending on when $n-k$ is odd or even.

Then the Thomas conjecture is as follows.

CONJECTURE (E. THOMAS). Suppose M satisfy conditions (A) and (B) with $k \leq \min(n/2, \rho(n)=2^b+8a-1)$ where $(n+1)=2^{4a+b}\times$(odd integer) and $0 \leq b \leq 3$. Then M admits a k-field without singularities.

1.1.2. We give a historical note on this particular conjecture. The reader who is interested in related conjectures should consult E. Thomas excellent survey article [30]. We digress a little because the conjecture should be viewed at together with his other conjectures.

In 1927, H. Hopf [14] proved a necessary and sufficient condition for a compact manifold M to admit a vector field, namely that M admits a 1-field iff the Euler characteristic of M vanishes. However it is easily seen by Poincare duality that if M is an odd-dimensional manifold then the Euler characteristic, $\chi(M) = 0$. After that little was done to prove the existence of 2-fields and 3-fields on manifolds in general except for low dimensional manifolds. In low dimensions we have the following.

Dimension 3. (Stiefel, 1936, [27]) Every orientable 3-manifold is parallelizable, i.e. the tangent bundle is trivial.

Dimension 5. (E. Thomas, 1968, [31])

1. Let Span(M) be the maximal number of linearly independent tangent vector fields on M. Then Span(M) ≥ 2 if $\hat{\chi}_2(M) = 0$ where $\hat{\chi}_2(M) = \sum_{i \leq n/2} \dim H_i(M; Z_2) \bmod 2$ is the semi-Kervaire characteristic.

2. Suppose $H^4(M;Z)$ has no element of order 2 and Span(M) ≥ 2 . Then Span(M) ≥ 3 iff M satisfies the following, namely, that there exists a class $x \in H^2(M;Z)$ such that $x^2 = P_1(M)$ (= the first Pontryagin class), $x \bmod 2 = w_2(M)$.

3. Suppose $H^4(M;Z)$ has no element of order 2. Then M is parallelizable iff $w_2(M)=0$, $P_1(M)=0$ and $\hat{\chi}_2(M)=0$.

Dimension 7. (E. Thomas, 1968, [32])

Every orientable manifold has Span ≥ 2 .

Suppose $H^4(M;Z)$ has no element of order two. Then,

1. If M is a spin manifold (i.e. $w_1(M)=w_2(M)=0$) $\mathrm{Span}(M) \geqslant 4$.

2. $\mathrm{Span}(M) \geqslant 5$ iff there exists a class $x \in H_2(M;Z)$ such that $x^2=P_1(M)$ and $x \bmod 2 = w_2(M)$.

3. M is parallelizable iff $w_2(M)=P_1(M)=0$.

So we observe from the above that in low dimensions something stronger is actually true. We refer the reader to Thomas [30] for conjectures that give a generalization of what may happen also in higher dimensions.

Since $\varrho(n)=1$ for $n \equiv 1$ or $5 \bmod 8$, $\varrho(n)=3$ for $n \equiv 3 \bmod 8$ and $\varrho(n) \geqslant 7$ for $n \equiv 7 \bmod 8$, the conjecture is true in dimensions $\leqslant 7$.

E. Thomas in 1967 disposed of the conjecture for the case $n \equiv 3 \bmod 4$ and $2 \leqslant k \leqslant 3$ in general. Massey [19] had shown that if $n \equiv 3 \bmod 4$ then $w_n(M)= w_{n-1}(M)=w_{n-2}(M)=0$ and that $\delta w_{n-3}(M)=0$ (where δ is the Bockstein coboundary operator associated with the exact sequence $0 \longrightarrow Z \longrightarrow Z \longrightarrow Z_2 \longrightarrow 0$) so that the primary obstructions for the existence of 2-fields and 3-fields vanish. We summarize Thomas work as follows including those on 4, 5 and 6 - fields.

Dimension of $M \equiv 3 \bmod 4 \geqslant 7$.

1. $\mathrm{Span}(M) \geqslant 3$.

2. If M is a spin manifold then $\mathrm{Span}(M) \geq 3$.

3. If $\dim M \equiv 7 \bmod 8 \geq 15$ and if for $4 \leq k \leq 6$ M is $(k-2)$-connected mod 2 and that $w_4(M)=0$, then $\mathrm{Span}(M) \geq k$.

The methods used in solving (1), (2) and (3) above involve identifying the obstructions via twisted secondary cohomology operations using a ' generating class ' theorem. Observations were then made that they reduce to Adams-Maunder operations [21] and finally they were shown to be zero. In Thomas [32] use is made of secondary cohomology operations on the Thom class of the tangent bundle to identify the final obstruction. The proof there consists of showing that the operation is zero modulo zero terminacy.

Atiyah-Dupont [3] has proved a much stronger result, namely , that for every M of dimension $\equiv 3 \bmod 4 \geq 7$ $\mathrm{Span}(M) \geq 3$. Their method uses real K-theory of stunted projective spaces. According to Massey [19] , $\delta w_{n-3}(M)=0$. Thus there exists a 3-fields on M with finite singularities. They identified the local obstructions that lie in $\pi_{n-1}(V_{n,3})$, via a construction in $KR^1(P_3,P_0)$ by $\theta : \pi_{n-1}(V_{n,3}) \longrightarrow KR^1(P_3,P_0)$ which was shown to be an isomorphism ($V_{n,k}$ is the Stiefel manifold of k-frame in n dimensional real space). It was shown that the image is given by the index of an element, constructed by using any 3-fields with finite singularities, in $KR^1(iTM \times (P_3,P_0))$. They observed that it actually comes from

$KR^1(P_\infty, P_0)$ which is zero and hence the existence of a 3-fields without singularities.

1.1.3. Definition. Suppose $F \longrightarrow E \xrightarrow{\pi} B$ is a fibration with the fibre, F, $(n-1)$-connected and suppose for convenience that B is simply connected as we shall be dealing only with fibrations with the base space simply connected. Let

$$1 < n(1) < n(2) < \cdots < n(j) < \cdots$$

be such that $\pi_{n(i)}(F) \neq 0$ for all i and $\pi_j(F)=0$ if $j \neq n(i)$ for some i. Then a Moore-Postnikov tower is a sequence of principal fibrations, $\{ p_i : E^i \longrightarrow E^{i-1} \}$ and maps $\{ q_i : E \longrightarrow E^i \}$ such that

(i) $p_i \circ q_i = q_{i-1}$, $E^0 = B$, $q_0 = \pi$;

(ii) the fibre of q_i, F^i, satisfies

$$\pi_j(F^i) \cong \begin{cases} \pi_j(F), & j > n(i), \\ 0, & j \leq n(i) \end{cases} ; \quad \text{and}$$

(iii) $p_i : E^i \longrightarrow E^{i-1}$ is induced by $k^i = \tau(s_{i-1}) \in H^{n(i)+1}(E^{i-1}; \pi_{n(i)})$ where $\pi_{n(i)} = \pi_{n(i)}(F)$, s_{i-1} is the fundamental class of F^{i-1} and τ is the transgression in the fibre space, $F^{i-1} \longrightarrow E \longrightarrow E^{i-1}$.

The classes $\{ k^i \}_i$ are called the Moore-Postnikov invariants or simply the k-invariants for the tower.

(1.1.4)
$$\begin{tikzcd}
 & E^i \arrow[d] & \\
 & E^{i-1} \arrow[r, "k^i"] \arrow[d] & K(\pi_{n(i)}, n(i)+1) \\
E \arrow[uur, "q^i"] \arrow[ur, "q^{i-1}"] \arrow[dr] \arrow[ddr, "\pi"'] & \vdots \arrow[d] & \\
 & E^1 \arrow[r, "k^2"] \arrow[d] & K(\pi_{n(2)}, n(2)+1) \\
 & B \arrow[r, "k^1"] & K(\pi_{n(1)}, n(1)+1)
\end{tikzcd}$$

1.1.5. Definition. Let $F \longrightarrow E \longrightarrow B$ be a fibration with B simply connected and $\pi_i(F)$ be finitely generated for each i. Consider the Moore-Postnikov tower for the fibration as given by Definition 1.1.3 and we use the notation of 1.1.3 in what follows.

Suppose X is a space and $\xi : X \longrightarrow B$ is a map such that $\xi^*(k^1)=0$. Then ξ lifts to E^1 and the obstruction to lifting ξ to E^2 is the <u>k-invariant for the lifting</u> and is the set

$$k^2(\xi) = \{ f^*(k^2) \mid f : X \longrightarrow E^1 \text{ a lifting of } \xi \text{ to } E^1 \} .$$

If $n(2)+1 < n(1)+\mathrm{Conn}(B)$, then $k^2(\xi)$ is a coset modulo

$$\Gamma_1 = \bigcup_h \{ (i_1 \circ h)^*(k^2) \mid h : X \longrightarrow \Omega C_1 \}$$

where $C_1 = K(\pi_{n(1)}, n(1)+1)$ and $i_1 : \Omega C_1 \longrightarrow E^1$ is the inclusion of the fibre in the total space. $k^{i+1}(\xi)$ is defined if $\{0\} \in k^i(\xi)$. If $n(t)+1 < n(t-1)+\mathrm{Conn}(B)$ for $2 \leq t \leq i$, then $k^i(\xi)$ † is a coset modulo

(1.1.6) $$\Gamma_{i-i} = \bigcup_h \{ g^*(k^{i+1}) \mid g : X \longrightarrow E^i \text{ a lifting of } X \xrightarrow{h} \Omega C_i \longrightarrow E^1 \} .$$

† In general the structure of $k^i(\xi)$ is either unknown or rather complicated. See Table 1.2.8 for example.

where the union is taken over all $h: M \longrightarrow \Omega C_1$ such that $(i_1 \circ h)^*(k^2) = 0$ and $\{0\} \in k^j(i_1 \circ h)$ for $j \leq i$.

1.1.7. Suppose we have a Moore-Postnikov tower for the fibration, $V_{n,k} \longrightarrow BSO_{n-k} \longrightarrow BSO_n$ for $n > k > 2$ and $k < n/2$ where $V_{n,k}$ is the Stiefel manifold of k-frame in n-dimensional real space. Let M be a (k-2)-connected n-dimensional smooth , orientable, closed and connected manifold. Then Poincaré duality and the universal coefficient theorem apply to give that $H^t(M;R) \cong 0$ for $n-k+2 \leq t \leq n-1$. Suppose $\delta w_{n-k}(M)=0$ if n-k is even or $w_{n-k+1}(M) = 0$ if n-k is odd; then the classifying map, $\tau : M \longrightarrow BSO_n$, for the tangent bundle of M lifts to E^1 and by the remark above it must lift to E^j where

$$n(j)+1 \leq n-1 < n(j+1)+1.$$

Note that $j \leq k-1$. $k^{j+1}(\tau)$ is the final obstruction and τ lifts to BSO_{n-k} iff $\{0\} \in k^{j+1}(\tau)$. (This is also true for a map, $\xi : X \longrightarrow BSO_n$, from a CW complex of dimension n to BSO_n, where X is (k-2)-connected and $\xi^*(k^1) = 0$.)

We call this the <u>single obstruction</u> to <u>lifting</u>. If $k > 5$ it is a coset modulo

$$\Gamma_{k-1} = \cup \{ g^*(k^{j+1}) \mid g : M \longrightarrow E^j, \text{ a lifting of } M \xrightarrow{h} \Omega C_1 \longrightarrow E^1 \}$$

where the union is taken over all $h : M \longrightarrow \Omega C_1$.† Usually $n(j+1)+1 = n$ and the single obstruction is non-trivial.

† This follows from the connectivity condition on M. The reason for calling $k^{j+1}(\tau)$ to be the single obstruction will be clearer after §1.2.

1.1.8. Let n and k be positive integers such that $n > k > 2$ and $k < n/2$. Using Serre $\mathcal{C}$ theory we see that if $n-k$ is odd and p is an odd prime then

$$
{}_p\pi_i(V_{n,k}) \cong \begin{cases} 0 \text{ for } n \text{ odd and } i < 2(n-k)+2p-2 \\ 0 \text{ for } n \text{ even and } i < \min(\,n-1, 2(n-k)+1)+2p-3\,. \end{cases}
$$

therefore through dimension n the question of the odd primary k-invariants occuring does not arise and we can modify the tower to decompose each fibration, $E^i \longrightarrow E^{i-1}$ where $i \leq j+1$ and $n(j)+1 \leq n-1 < n(j+1)+1$, into a sequence of principal fibrations with mod 2 k-invariants. This is due to Mahowald-Gitler [12] and is sketched in §2

But if n-k is even ${}_p\pi_{(n-k)+2p-3}(V_{n,k}) \cong Z_p$ which implies that we may have mod p k-invariants possibly occuring in dimensions $\leq n$. From the fibration,

$$V_{n-k+1,1} \longrightarrow V_{n,k} \longrightarrow V_{n,k-1},$$

we get the exact sequence of homotopy groups

$$\pi_{n-1}(V_{n-k+1,1}) \longrightarrow \pi_{n-1}(V_{n,k}) \longrightarrow \pi_{n-1}(V_{n,k-1})\,.$$

Since we know $\pi_{n-1}(V_{n,k-1})$ is 2-primary and that, according to H. Toda [37], ${}_p\pi_{n-k+2t(p-1)-1}(V_{n-k+1,1}) \cong Z_p$ for $1 \leq t \leq p-1$ so that for n odd and $k \leq 9$ there is no odd p-primary k-invariants occuring in dimension n. However, if $k = 11$ then we have a mod 3 k-invariants occuring in dimension n for n odd.

REMARK. According to Mahowald-Copeland [18] , the odd obstructions in dimensions $\leq 2(n-k)-1$ always vanish for any $\xi : X \longrightarrow BSO_n$ whose primary obstruction vanishes. Thus , if the dimension of X is less than or equal to $2(n-k)-1$, then the mod 2 modified Postnikov tower is the relevant one to study the lifting problem of ξ to BSO_{n-k}. We refer the reader to the above paper.

1.1.9. Below we list some of the homotopy groups to be killed (see G. Paechter [24] , Hoo-Mahowald [13]). We assume $k > 6$ and $k < n/2$.

$n \equiv 3(8)$ TABLE 1.1.10. of $\pi_i(V_{n,k})$, $i \leq n-k+5$ and $k \equiv t(8)$

t \ i	n-k	n-k+1	n-k+2	n-k+3	n-k+4	n-k+5
1	Z	Z_4	0	Z_2	Z_2	Z_{16}
5	Z	Z_4	0	Z_2	0	Z_8
3	Z	Z_2+Z_2	Z_2+Z_2	$Z_{24}+Z_8$	Z_2	0
7	Z	Z_2+Z_2	Z_2+Z_2	Z_4+Z_{48}	Z_2	0
2	Z_2	Z_2	Z_8	Z_2	0	Z_2
6	Z_2	Z_2	Z_8	Z_2	0	0
4	Z_2	0	Z_2	Z_2	Z_8	Z_2
8	Z_2	0	Z_2	Z_2+Z_2	Z_{16}	Z_2

$n \equiv 7(8)$ TABLE 1.1.11. of $\pi_i(V_{n,k})$, $i \leq n-k+5$ and $k \equiv t$ (8)

t \ i	n-k	n-k+1	n-k+2	n-k+3	n-k+4	n-k+5	n-k+6
1	Z	Z_4	0	Z_{12}	0	Z_8	
5	Z	Z_4	0	Z_{12}	Z_2	Z_{16}	
3	Z	Z_2+Z_2	Z_2+Z_2	Z_4+Z_{48}	Z_2	0	
7	Z	Z_2+Z_2	Z_2+Z_2	Z_8+Z_{24}	Z_2	0	Z_2+Z_2 $k\equiv 7$
2	Z_2	Z_2	Z_8	Z_2	0	0	
6	Z_2	Z_2	Z_8	Z_2	0	Z_2	
4	Z_2	0	Z_2	Z_2+Z_2	Z_{16}	Z_2	
8	Z_2	0	Z_2	Z_2	Z_8	Z_2	

§1.2. The Modified Postnikov Towers of M. Mahowald

1.2.1. Definition. A t-modified Postnikov tower (t-MPT) for the fibration, $F \longrightarrow E \xrightarrow{\pi} B$, is a sequence of fibrations,

$$E^m \xrightarrow{q_m} E^{m-1} \longrightarrow \cdots \longrightarrow E^1 \xrightarrow{q_1} B,$$

and maps, $\{ p_i : E \longrightarrow E^i \}_{i \leq m}$, such that

(1) $q_i \circ p_i = p_{i-1}$, $E^0 = B$, $p_0 = \pi$;

(2) the fibre of q_i, C_i, is a product of Eilenberg-MacLane spaces, $K(\pi, k)$ where $\pi = Z$ or Z_p, p is a prime and $k < t$; and (3) the fibre of p_i, F^i is $t(i)$-connected where $t(m) \geq t-1$ and if $\iota : F^{i+1} \subset F^i$ is the inclusion then $\iota^* : H^s(F^i) \longrightarrow H^s(F^{i+1})$ is trivial for $s \leq t$.

By remark 1.1.9. we shall ignore the mod p k-invariants for odd prime p and construct a n-MPT following Mahowald [17]

(We can in fact obtain a MPT through dimension $2(n-k)$ for the fibration, $V_{n,k} \longrightarrow BSO_{n-k} \longrightarrow BSO_n$). First we recall the the following observation of F. Adams:

1.2.2. Theorem (Adams [2], pp. 62). Let A_r = algebra generated by $\{ Sq^1, Sq^2, \ldots, Sq^{2^r} \}$ when r is finite; $A_\infty = \mathfrak{A}(2)$ the mod 2 Steenrod algebra and A_0 is the exterior algebra generated by Sq^1. Suppose L is a left A_r-module such that L is free left module over A_0 and $L_t = 0$ for $t < j$. Then $\mathrm{Tor}^{A_r}_{s,t}(Z_2, L)$ and $\mathrm{Ext}^{s,t}_{A_r}(L, Z_2)$ are zero if $t < j + T(s)$ where $T(s)$ is the numerical function defined by

$$T(4q) = 12q \ ,$$

$$T(4q+1) = 12q+2 \ ,$$

$$T(4q+2) = 12q+4 \text{ and}$$

$$T(4q+3) = 12q+7 \ ;$$

and that the homomorphisms (change of ring)induced by $A_\rho \longrightarrow A_r$,

$$\iota_* : \mathrm{Tor}^{A_\rho}_{s,t}(Z_2, L) \longrightarrow \mathrm{Tor}^{A_r}_{s,t}(Z_2, L) \text{ and}$$

$$\iota^* : \mathrm{Ext}^{s,t}_{A_\rho}(L, Z_2) \longleftarrow \mathrm{Ext}^{s,t}_{A_r}(L, Z_2) \quad ,$$

are isomorphisms if $0 < \rho < r$, $s \geq 1$ and $t < j + T(s-1) + 2^{\rho+1}$.

We shall need the following facts:

1.2.3. FACTS. Let $k < n/2$ for convenience.

(1) $H^*(V_{n,k}; Z)$ has no odd torsions;

(2) $H^{n-k}(V_{n,k}; Z) \cong \begin{cases} Z & \text{if } k = 1 \text{ or } n-k \equiv 0 \bmod 2 \ , \\ 0 & \text{otherwise} \ ; \end{cases}$

(3) $H^{n-1}(V_{n,k}; Z) \cong \begin{cases} Z_2 & n \equiv 1 \bmod 2 \ , \\ Z & n \equiv 0 \bmod 2 \ ; \end{cases}$ and

(4) $H^i(V_{n,k};Z) \cong 0 \quad n-1 < i < 2(n-k)$.

We shall now proceed to give a sketch of the construction of a t-MPT for the fibration, $V_{n,k} \longrightarrow BSO_{n-k} \longrightarrow BSO_n$, for $t < 2(n-k)$ which is due to Mahowald and Gitler [12]; we refer the reader to their paper for details. The main idea is to take an $\mathcal{A}(2)$-free resolution of $H^*(V_{n,k})$ through dimension $t < 2(n-k)$,

(1.2.4.) $$0 \longleftarrow H^*(V_{n,k}) \longleftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \longleftarrow \cdots ,$$

and make the following observation:

Associated with such a resolution is a geometrical realization,

(1.2.5) $$F^{s+1} \xrightarrow{h_{s+1}} F^s \longrightarrow \cdots \longrightarrow F^2 \xrightarrow{h_2} F^1 \xrightarrow{h_1} V_{n,k} .$$

Then associated with such a realization is a decomposition of the fibration, $BSO_{n-k} \longrightarrow BSO_n$. To see how one can construct such a resolution, let us suppose the following portion of the MPT has been constructed:

(1.2.6)
$$\begin{tikzcd}
F^s \arrow[r, "h_s"] & F^{s-1} \arrow[r] \arrow[d, "g_{s-1}"'] & \cdots \arrow[r] & F^1 \arrow[r, "h_1"] & V_{n,k} \arrow[r] \arrow[dddd, "g_1"] & BSO_{n-k} \arrow[d, "p_s"] \arrow[dd, bend left, "p_{s-1}"] \\
& G_{s-1} \arrow[rrrr] & & & & E^s \arrow[d, "q_s"] \\
& & & & & E^{s-1} \arrow[d] \\
& & & & & \vdots \arrow[d] \\
& & & & G_0 \arrow[r] & E^1 \arrow[d, "q_1"] \\
& & & & & BSO_n
\end{tikzcd}$$

where $\{F^i \xrightarrow{h_i} F^{i-1} \xrightarrow{g_{i-1}} G_{i-1}\}_{i \leq s}$ and $\{G_{i-1} \longrightarrow E^i \longrightarrow E^{i-1}\}_{i \leq s}$ are principal fibrations, $H^*(G_i) \cong C_i$ through dimensions $\leq$

$2(n-k)-1$ as $\mathcal{A}(2)$-modules for $i \leq s-1$, and $g_i^*: H^*(G_i) \longrightarrow H^*(F^i)$ is an epimorphism in dimensions $\leq t$. Then according to E. Thomas [28] the sequence

$$H^*(E^s) \xrightarrow{\nu_s} H^*(BSO_{n-k} \times G_{s-1}) \xrightarrow{\tau_1} H^*(E^{s-1})$$

is exact at least in dimensions $\leq 2(n-k)$ and ν_s^* is monomorphic in dimensions $\leq t$. Using this and the fact that d_s determines a set of $\mathcal{A}(2)$-free generators of $\mathrm{Ker}(g_{s-1}^*)$ through dimensions $\leq t$ from which we can associate a generalized Eilenberg-MacLane space G_s, we see that there is determined a set of $\mathcal{A}(2)$-free generators of $\mathrm{Ker}(p_s^*)$ through dimensions $\leq t$ and so a principal fibration, $G_s \longrightarrow E^{s+1} \xrightarrow{q_{s+1}} E^s$, and a lifting, $p_{s+1}: BSO_{n-k} \longrightarrow E^{s+1}$, of p_s to E^{s+1}. Note that in fact d_s determines a map $g_s: F^s \longrightarrow G_s$ and the fibre of g_s, F^{s+1}, can be regarded as the fibre of p_{s+1}. We have also that g_s^* is an epimorphism in dimensions $\leq t$ and that $H^*(G_s) \cong C_s$ as $\mathcal{A}(2)$-modules through dimensions $\leq 2(n-k)-1$. It is clear how one can construct E^1 and so we can inductively construct E^i until we obtain a t-MPT. Using Theorem 1.2.2 we see that F^s is min(t, $n-k+T(s)-s-1$)-connected modulo odd torsions (see Proposition 7.12 of Mahowald-Gitler [12]) and so a t-MPT is obtainable in a finite number of stages.

Mahowald [17] had given some indication as to how one can compute the first k-invariants. For example, if $n \equiv 7 \bmod 8$, $k \geq 7$ and $k \leq n/2$ then the first k-invariants for $(n-k+4)$-MPT is given by Table 1.2.7 below.

TABLE 1.2.7. The 1^{st} k-invariants for $BSO_{n-k} \longrightarrow BSO_n$

$n \equiv 7 \bmod 8, k \geq 7$ $k < n/2$.

Dimension	n-k+1	n-k+2	n-k+3	n-k+4
$k \equiv \bmod 8$				
1	δw_{n-k}	w_{n-k+2}	0	0
5	δw_{n-k}	w_{n-k+2}	0	0
3	δw_{n-k}	w_{n-k+2}	0	w_{n-k+4}
7	δw_{n-k}	w_{n-k+2}	0	w_{n-k+4}
2, 6	w_{n-k+1}	0	w_{n-k+3}	0
4, 0	w_{n-k+1}	0	0	0

The stage 2 k-invariants are then determined as was outlined before. We list the result of the computation giving the defining relations and the names of the k-invariants in Table 1.2.8. We let H^* denote mod 2 cohomology and adopt the following notation:

$A_i = (Sq^2+w_2.)H^i$;

$B_i = (Sq^2+w_2.)Sq^1H^i$;

$D_i = (Sq^4+w_4.)H^i$ and respectively in integral cohomology denoted by H_o

$A_i^0 = (Sq^2+w_2.)H_o^i$;

$B_i^0 = (Sq^2+w_2.)Sq^1H_o^i$; and

$D_i^0 = (Sq^4+w_4.)H_o^i$.

TABLE 1.2.8 The 2nd Stage k-invariants for $BSO_{n-k} \longrightarrow BSO_n$

$n \equiv 7 \bmod 8,\ k \geqslant 7,\ k < n/2.$

k≡(8)	Dim	k-invariant	Defining Relation	Indeterminacy
6	n-k+2	k_1^2	$(Sq^2+w_2\cdot)w_{n-k+1}=0$	A_{n-k}
	n-k+3	k_2^2	$(Sq^2+w_2\cdot)Sq^1 w_{n-k+1}+ Sq^1 w_{n-k+3}=0$	$B_{n-k}\oplus Sq^1 H^{n-k+2}$
	n-k+4	k_3^2	$(Sq^4+w_4\cdot)w_{n-k+1}+ w_2\cdot w_{n-k+3}=0$	$D_{n-k}\oplus w_2\cdot H^{n-k+2}$
2	n-k+2	k_1^2	$(Sq^2+w_2\cdot)w_{n-k+1}=0$	A_{n-k}
	n-k+3	k_2^2	as for $k \equiv 6(8)$	as for $k \equiv 6(8)$
	n-k+4	k_3^2	$(Sq^4+w_4\cdot)w_{n-k+1}+ Sq^2 w_{n-k+3}=0$	$D_{n-k}\oplus Sq^2 H^{n-k+2}$
4	n-k+3	k_1^2	$(Sq^2+w_2\cdot)Sq^1 w_{n-k+1}=0$	B_{n-k}
	n-k+4	k_2^2	$(Sq^4+w_4\cdot)w_{n-k+1}=0$	D_{n-k}
0	n-k+3	k_1^2	$(Sq^2+w_2\cdot)Sq^1 w_{n-k+1}=0$	B_{n-k}
7	n-k+2	k_1^2	$(Sq^2+w_2\cdot)\delta w_{n-k}=0$	A_{n-k}^0
	n-k+3	k_2^2	$(Sq^2+w_2\cdot)w_{n-k+2}=0$	A_{n-k+1}
	n-k+4	k_3^2	$(Sq^4+w_4\cdot)\delta w_{n-k}+ Sq^1(w_2\cdot w_{n-k+2})=0$	$D_{n-k}^0\oplus(w_3\cdot+w_2\cdot Sq^1)H^{n-k+1}$
	n-k+4	k_4^2	$Sq^1 w_{n-k+4}+ (Sq^2+w_2\cdot)Sq^1 w_{n-k+2}=0$	$B_{n-k+1}\oplus Sq^1 H^{n-k+3}$
3	n-k+4	k_1^2	$(Sq^4+w_4\cdot)\delta w_{n-k}+Sq^1(w_2\cdot w_{n-k+2}+w_{n-k+4})=0$	$D_{n-k}^0\oplus Sq^1 H^{n-k+3}\oplus(w_3\cdot+w_2\cdot Sq^1)H^{n-k+1}$
	n-k+4	k_2^2	$(Sq^4+w_4\cdot)\delta w_{n-k}+(Sq^2Sq^1+w_3\cdot)w_{n-k+2}=0$	$D_{n-k}^0\oplus(Sq^2Sq^1+w_3\cdot)H^{n-k+1}$

TABLE 1.2.8 The 2nd Stage k-invariants for $BSO_{n-k} \longrightarrow BSO_n$

$n \equiv 7 \bmod 8,\ k \geqslant 7,\ k < n/2.$

$k \equiv (8)$	Dim	k-invariant	Defining Relation	Indeterminacy
6	$n-k+2$	k_1^2	$(Sq^2 + w_2 \cdot) w_{n-k+1} = 0$	A_{n-k}
	$n-k+3$	k_2^2	$(Sq^2 + w_2 \cdot) Sq^1 w_{n-k+1} + Sq^1 w_{n-k+3} = 0$	$B_{n-k} \oplus Sq^1 H^{n-k+2}$
	$n-k+4$	k_3^2	$(Sq^4 + w_4 \cdot) w_{n-k+1} + w_2 \cdot w_{n-k+3} = 0$	$D_{n-k} \oplus w_2 \cdot H^{n-k+2}$
2	$n-k+2$	k_1^2	$(Sq^2 + w_2 \cdot) w_{n-k+1} = 0$	A_{n-k}
	$n-k+3$	k_2^2	as for $k \equiv 6(8)$	as for $k \equiv 6(8)$
	$n-k+4$	k_3^2	$(Sq^4 + w_4 \cdot) w_{n-k+1} + Sq^2 w_{n-k+3} = 0$	$D_{n-k} \oplus Sq^2 H^{n-k+2}$
4	$n-k+3$	k_1^2	$(Sq^2 + w_2 \cdot) Sq^1 w_{n-k+1} = 0$	B_{n-k}
	$n-k+4$	k_2^2	$(Sq^4 + w_4 \cdot) w_{n-k+1} = 0$	D_{n-k}
0	$n-k+3$	k_1^2	$(Sq^2 + w_2 \cdot) Sq^1 w_{n-k+1} = 0$	B_{n-k}
7	$n-k+2$	k_1^2	$(Sq^2 + w_2 \cdot) \delta w_{n-k} = 0$	A^0_{n-k}
	$n-k+3$	k_2^2	$(Sq^2 + w_2 \cdot) w_{n-k+2} = 0$	A_{n-k+1}
	$n-k+4$	k_3^2	$(Sq^4 + w_4 \cdot) \delta w_{n-k} + Sq^1 (w_2 \cdot w_{n-k+2}) = 0$	$D^0_{n-k} \oplus (w_3 \cdot + w_2 \cdot Sq^1) H^{n-k+1}$
	$n-k+4$	k_4^2	$Sq^1 w_{n-k+4} + (Sq^2 + w_2 \cdot) Sq^1 w_{n-k+2} = 0$	$B_{n-k+1} \oplus Sq^1 H^{n-k+3}$
3	$n-k+4$	k_1^2	$(Sq^4 + w_4 \cdot) \delta w_{n-k} + Sq^1 (w_2 \cdot w_{n-k+2} + w_{n-k+4}) = 0$	$D^0_{n-k} \oplus Sq^1 H^{n-k+3} \oplus (w_3 \cdot + w_2 \cdot Sq^1) H^{n-k+1}$
	$n-k+4$	k_2^2	$(Sq^4 + w_4 \cdot) \delta w_{n-k} + (Sq^2 Sq^1 + w_3 \cdot) w_{n-k+2} = 0$	$D^0_{n-k} \oplus (Sq^2 Sq^1 + w_3 \cdot) H^{n-k+1}$

TABLE 1.2.8 CTD.

$k \equiv (8)$	Dim	k-invariant	Defining Relation	Indeterminacy
5	n-k+4	k_1^2	$(Sq^4+w_4.)\delta w_{n-k}+ w_3.w_{n-k+2} = 0$	$D^0_{n-k} \oplus w_3.H^{n-k+1}$
1	n-k+4	k_1^2	$(Sq^4+w_4.)\delta w_{n-k} + (Sq^3+w_2.Sq^1)w_{n-k+2} = 0$	$D^0_{n-k} \oplus (Sq^3+w_2.Sq^1)H^{n-k+1}$

1.2.9. Before we go on to investigate the fibration, $V_{n,7} \longrightarrow BSO_{n-7} \xrightarrow{\pi} BSO_n$, for $n \equiv 7 \bmod 8$, $n > 7$, we list the following facts from which we deduce that $\mathrm{Ker}(\pi^*)$ in integral cohomology is 2-torsion in the range we are considering.

(1). $H^*(BSO_{2q+1};Z) \cong Z[P_1, P_2, \ldots, P_q] \oplus T$, where T is 2-torsion;

(2) $H^*(BSO_{2q};Z) \cong Z[P_1, P_2, \ldots, P_q, \chi_{2q}] \oplus T$ modulo the ideal generated by $(P_q - \chi^2_{2q})$; let p be an odd prime, then

(3) $H^*(BSO_{2q+1};Z_p) \cong Z_p[P_1, P_2, \ldots, P_q]$;

(4) $H^*(BSO_{2q};Z_p) \cong Z_p[P_1, P_2, \ldots, P_q, \chi_{2q}]$ modulo the ideal generated by $(P_q - \chi^2_{2q})$;

(5) $H^*(BSO_s:Z_2) \cong Z_2[w_2, w_3, \ldots, w_s]$ and

(6) the Kernel of Sq^1 in $H^*(BSO_s:Z_2)$ consists of integral cohomology classes reduced mod 2.

We have adopted the standard notation χ_s for the Euler class.

1.2.10. The Fibration, $V_{n,7} \longrightarrow BSpin_{n-7} \xrightarrow{\pi} BSpin_n$, for

$n \equiv 7 \bmod 8$ and $n > 15$.

Since the calculation is fairly routine but tedious we shall do the computation for a 3-stage (n-) Postnikov tower through dimension n.

It is easily seen that the k-invariants for the first stage are given by

$$k_1^1 = \delta w_{n-7}\ ,\ k_2^1 = w_{n-5} \text{ and } k_3^1 = w_{n-3}$$

(by using Wu's formula and the result can be found in Table 1.2.7). We shall proceed to obtain the k-invariants for the second stage to give an indication to the computation in Table 1.2.8. Consider the following commutative diagram:

$$\begin{tikzcd} BSpin_{n-7}\times\Omega C_1 \arrow[r, "\nu_1"] \arrow[d] & E^1 \arrow[d, "q_1"] & \\ BSpin_{n-7} \arrow[u, bend left, "s_1"] \arrow[ur, "p_1"] \arrow[r, "\pi"] & BSpin_n \arrow[r] & K^*_{n-6}\times K_{n-5}\times K_{n-3} = C_1 \end{tikzcd} \qquad (1.2.11)$$

where $s_1 : BSpin_{n-7} \longrightarrow BSpin_{n-7}\times\Omega C_1$ is the trivial section. E. Thomas [28] derived the following exact sequence,

$$0 \longrightarrow H^t(E^1) \xrightarrow{\nu_1^*} H^t(BSpin_{n-7}\times\Omega C_1) \xrightarrow{\tau_1} H^{t+1}(BSpin_n)\ ,$$

for $t \leq 2(n-7)-1$ where ν_1^* is monomorphic in dimensions $\leq 2(n-7)$. We identify the kernel of p_1^* in dimensions $\leq n$ as the kernel of s_1^* in kernel of τ_1. We use the fact that τ_1 is a $H^*(BSpin_n)$-homomorphism. The result is tabulated below:

TABLE 1.2.12. Ker (p_1^*)

Dimension	$\mathrm{Ker}(p_1^*) \cong \mathrm{Ker}(\tau_1^*) \cap \mathrm{Ker}(s_1^*)$
$\leq$ n-6	0
n-5	$1\otimes Sq^2\iota^*_{n-7}$
n-4	$1\otimes Sq^3\iota^*_{n-7}$, $1\otimes Sq^2\iota_{n-6}$
n-3	$1\otimes Sq^3\iota_{n-6}$, $1\otimes Sq^4\iota^*_{n-7}+w_4\otimes\iota^*_{n-7}$, $1\otimes Sq^2Sq^1\iota_{n-6}+1\otimes Sq^1\iota_{n-4}$
n-2	$1\otimes Sq^5\iota^*_{n-7}$, $1\otimes Sq^3Sq^1\iota_{n-6}$, $1\otimes Sq^4\iota_{n-6}+w_4\otimes\iota_{n-6}$
n-1	$1\otimes Sq^4Sq^2\iota^*_{n-7}$, $w_4\otimes Sq^2\iota^*_{n-7}$, $1\otimes Sq^6\iota^*_{n-7}+w_6\otimes\iota^*_{n-7}$, $1\otimes Sq^5\iota_{n-6}+w_4\otimes Sq^1\iota_{n-6}$
n	$1\otimes Sq^5Sq^2\iota^*_{n-7}$, $w_4\otimes Sq^3\iota^*_{n-7}$, $1\otimes Sq^4Sq^2\iota_{n-6}$, $w_4\otimes Sq^2\iota_{n-6}$, $1\otimes Sq^3Sq^1\iota_{n-4}$, $1\otimes Sq^7\iota^*_{n-7}+w_7\otimes\iota^*_{n-7}$, $1\otimes Sq^6\iota_{n-6}+w_6\otimes\iota_{n-6}$, $1\otimes Sq^4\iota_{n-4}+w_4\otimes\iota_{n-4}$

What are given in the column on the right are the Z_2-basis of $\mathrm{Ker}(p_1^*)$ in dimensions $\leq n$.

The $\mathcal{A}(2)(BSpin_n)$ generators through dimensions $\leq n$ obtained from Table 1.2.12 are given by Table 1.2.13 where by abuse of notation we have identified the k-invariants with their monomorphic images in $H^*(BSpin_{n-7}\times\Omega C_1)$.

TABLE 1.2.13 The k-invariants for the 2^{nd} Stage of an n-MPT for $V_{n,7} \longrightarrow BSpin_{n-7} \longrightarrow BSpin_n$ and $n \equiv 7(8) > 15$.

Dimension	k-invariant = $\mathcal{A}(2)(BSpin_n)$-generator of $Ker(p_1^*)$ through dimensions $\leq n$
n-5	$k_1^2 = 1\otimes Sq^2\iota^*_{n-7}$
n-4	$k_2^2 = 1\otimes Sq^2\iota_{n-6}$
n-3	$k_3^2 = 1\otimes Sq^4\iota^*_{n-7}+w_4\otimes\iota^*_{n-7}$
n-3	$k_4^2 = 1\otimes Sq^2Sq^1\iota_{n-6}+1\otimes Sq^1\iota_{n-4}$
n-2	$k_5^2 = 1\otimes Sq^4\iota_{n-6}+w_4\otimes\iota_{n-6}$
n	$k_6^2 = 1\otimes Sq^4\iota_{n-4}+w_4\otimes\iota_{n-4}$

To find the k-invariants for the third stage we proceed as before obtaining the following commutative diagram:

$$
(1.2.14)\qquad
\begin{tikzcd}
F^2 \arrow[r] & F^1 \arrow[r] \arrow[d] & V_{n,7} \arrow[r] \arrow[dd] & BSpin_{n-7} \arrow[d, "p_2"] \arrow[dd, bend left, "p_1"] & \\
& \Omega C_2 \arrow[rr] & & E^2 \arrow[d, "q_2"] & \\
& & \Omega C_1 \arrow[r] & E^1 \arrow[r] \arrow[d, "q_1"] & C_2 = K_{n-5}\times K_{n-4}\times K^1_{n-3}\times K^2_{n-3}\times K_{n-2}\times K_n \\
& & & BSpin_n \arrow[r] & C_1 = K^*_{n-6}\times K_{n-5}\times K_{n-3}
\end{tikzcd}
$$

We have in dimensions $\leq$ n the following exact sequence of Thomas,

$$0 \longrightarrow H^*(E^2) \xrightarrow{\nu_2^*} H^*(BSpin_{n-7}\times\Omega C_2) \xrightarrow{\tau_1} H^{*+1}(E^1) \quad .$$

We use ν_1^* to determine the kernel of $\tau_1^* = Ker(\nu_1\tau_1)$ and hence the image of ν_2^* and compute the kernel of the homomorphism induced by the trivial section of the trivial bundle induced from the 2^{nd} stage of the n-MPT by p_1, in $Im(\nu_2^*)$. A set of $\mathcal{A}(2)(Bspin_n)$ generators then constitute the next k-invariants.

TABLE 1.2.15 Kernel (p_2^*) in dim $\leqslant$ n

Dim	Z_2-basis of $\mathrm{Ker}(p_2^*) = \mathrm{Ker}(s_2^*) \cap \mathrm{Ker}(\tau_1)$
$\leqslant$ n-5	0
n-4	$1 \otimes Sq^2 \iota_{n-6}$
n-3	$1 \otimes Sq^3 \iota_{n-6}$, $1 \otimes Sq^2 Sq^1 \iota_{n-6} + 1 \otimes Sq^1 \iota^1_{n-4}$, $1 \otimes Sq^2 \iota_{n-5} + 1 \otimes Sq^1 \iota^2_{n-4}$
n-2	$1 \otimes Sq^3 Sq^1 \iota_{n-6}$, $1 \otimes Sq^3 \iota_{n-5}$
n-1	$1 \otimes Sq^2 Sq^1 \iota^1_{n-4}$, $1 \otimes Sq^3 Sq^1 \iota_{n-5} + 1 \otimes Sq^2 Sq^1 \iota^2_{n-4}$ $1 \otimes Sq^2 Sq^3 \iota_{n-6}$
n	$1 \otimes Sq^5 Sq^1 \iota_{n-6}$, $1 \otimes Sq^4 Sq^2 \iota_{n-6}$, $1 \otimes (Sq^5 + Sq^4 Sq^1 \iota_{n-5})$, $1 \otimes Sq^3 Sq^1 \iota^2_{n-4}$, $1 \otimes Sq^3 Sq^1 \iota^1_{n-4}$, $w_4 \otimes Sq^2 \iota_{n-6}$, $1 \otimes Sq^4 \iota^1_{n-4} + w_4 \otimes \iota^1_{n-4} + 1 \otimes Sq^6 \iota_{n-6} + w_6 \otimes \iota_{n-6}$

TABLE 1.2.16 The k-invariants for the 3^{rd} stage of an n-MPT for $V_{n,7} \to BSpin_{n-7} \to BSpin_n$ $n \equiv 7(8) > 15$.

Dim	k-invariant
n-4	$k^3_1 = 1 \otimes Sq^2 \iota_{n-6}$
n-3	$k^3_2 = 1 \otimes Sq^2 Sq^1 \iota_{n-6} + 1 \otimes Sq^1 \iota^1_{n-4}$
n-3	$k^3_3 = 1 \otimes Sq^2 \iota_{n-5} + 1 \otimes Sq^1 \iota^2_{n-4}$
n	$k^3_4 = 1 \otimes Sq^4 \iota^1_{n-4} + w_4 \otimes \iota^1_{n-4} + 1 \otimes Sq^6 \iota_{n-6} + w_6 \otimes \iota_{n-6}$

Repeating in this fashion we obtain the following exact sequence in dimensions $\leqslant 2(n-7)-1$ where ν_3^* is a monomorphism in dimensions $\leqslant$ n :

$$0 \longrightarrow H^*(E^3) \xrightarrow{\nu_3^*} H^*(BSpin_{n-7}\times\Omega C_3) \xrightarrow{\tau_1} H^{*+1}(E^2)$$

where $C_3 \equiv K_{n-4}\times K^1_{n-3}\times K^2_{n-3}\times K_n$.

As before we use the homomorphism, $\nu_2^*: H^*(E^2)\longrightarrow H^*(BSpin_{n-7}\times\Omega C_2)$, to determine the image of ν_3^* and this was then used to compute the kernel of p_3^* in dimensions $\leqslant$ n. The k-invariants for the fourth stage n-MPT are given in Table 1.2.18 which follows.

TABLE 1.2.18 The k-invariants for the 4th Stage n-MPT for $V_{n,7}\longrightarrow BSpin_{n-7}\longrightarrow BSpin_n$ and $n\equiv 7(8) > 15$.

Dim	k-invariants
$\leqslant$ n-4	0
n-3	$k^4_1 = 1\otimes Sq^2\iota_{n-5}+1\otimes Sq^1\iota^1_{n-4}$
n-2	0
n-1	0
n-1	0

In summary, we have the following decomposition:

(1.2.19)
$$\begin{tikzcd}
F^4 \arrow[r] & F^3 \arrow[r] \arrow[d] & F^2 \arrow[r] \arrow[dd] & F^1 \arrow[r] \arrow[ddd] & V_{n,7} \arrow[r] \arrow[dddd] & BSpin_{n-7} \arrow[d] & \\
& \Omega C_4 \arrow[rrrr] & & & & E^4 \arrow[d] & \\
& & \Omega C_3 \arrow[rrr] & & & E^3 \arrow[r, "k^4"] \arrow[d] & C_4 \\
& & & \Omega C_2 \arrow[rr] & & E^2 \arrow[r, "k^3"] \arrow[d] & C_3 \\
& & & & \Omega C_1 \arrow[r] & E^1 \arrow[r, "k^2"] \arrow[d] & C_2 \\
& & & & & BSpin_n \arrow[r, "k^1"] & C_1
\end{tikzcd}$$

§1.3. Twisted Secondary Cohomology Operations and the Single Obstruction

1.3.1. The split tensor algebra (Massey-Peterson algebra).

If $\mathcal{A}_p$ is the mod p Steenrod algebra for p a prime, the split tensor algebra $\mathcal{A}_p(X)$ for a topological space X has as its underlying vector space, $H^*(X)\otimes\mathcal{A}_p$, and with multiplication as follows:

for $v\otimes\alpha$, $u\otimes\beta \in \mathcal{A}_p(X)$,

$$(v\otimes\alpha).(u\otimes\beta) = \sum_i (-1)^{|\alpha_i|.|u|}(v.\alpha_i u)\otimes\alpha_i'\beta ,$$

where $\psi(\alpha) = \sum_i \alpha_i\otimes\alpha_i'$ and $\psi:\mathcal{A}_p \longrightarrow \mathcal{A}_p\otimes\mathcal{A}_p$ is the diagonal homomorphism in $\mathcal{A}_p$.

It has the following properties:

(1) If $\xi : Y \longrightarrow X$ is a map from a space to another, then $H^*(Y)$ is a left $H^*(X)$-module via the action given by

for $b \in H^*(X)$ and $y \in H^*(Y)$ $b . y = \xi^*(b)\cup y$ where $\cup$ is the cup product in $H^*(Y)$ so that $H^*(Y)$ become an $\mathcal{A}_p(X)$-module via the action

$$(x\otimes a).y = x.a(y) = \xi^*(x)\cup a(y)$$

for $x\otimes a \in \mathcal{A}_p(X)$ and $y \in H^*(Y)$;

(2) if $\zeta : Y' \longrightarrow Y$ is a map between two spaces then $\zeta^*: H^*(Y) \longrightarrow H^*(Y')$ is a morphism of $\mathcal{A}_p(X)$-modules where Y is as in (1) above ; and

(3) obviously we have the natural map

$\xi^* : \mathcal{A}_p(X) \longrightarrow \mathcal{A}_p(Y)$ where ξ is as in (1) above.

1.3.2. Higher Order Twisted Cohomology Operations.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k \in \mathcal{A}_2(Y)$ such that $\alpha \cdot \beta = \Sigma\, \alpha_i \cdot \beta_i = 0$ is a relation of degree q+1 in $\mathcal{A}_2(Y)$. Given a map $\xi : X \longrightarrow Y$ and a cohomology class $u \in H^s(X,B)$, where (X,B) is a pair of spaces, represented by a map also denoted by u, $u: (X,B) \longrightarrow (K_s,*)$ .where * is a base point in K_s. Let $C = K^1 \times \ldots \times K^k$ where $K^i = K(Z_2, s+\deg\beta_i)$ and consider the principal fibration,

$$(E, \overline{Y}) \xrightarrow{p_1} Y\times(K_s, *) \xrightarrow{\beta.(1\otimes\iota_s)} (C, *) \quad ,$$

induced by $\beta.(1\otimes\iota_s)$ where ι_s is the fundamental class of $(K_s,*)$. Note that p_1 maps $\overline{Y}$ homeomorphically onto Y. We let $\mathcal{A}_2(Y)$ act on $H^*(\Omega C\times E, E)$ via the map,

$$\Omega C\times E \longrightarrow E \xrightarrow{p_1} Y\times K_s \longrightarrow Y$$

where $\Omega C\times E \longrightarrow E$ and $Y\times K_s \longrightarrow Y$ are the respective projection. We have the following exact sequence:

$$0 \longrightarrow H^*(\Omega C\times E,\ E) \xrightarrow{j^*} H^*(\Omega C\times E,\ \overline{Y}) \xrightarrow{i^*} H^*(E,\ \overline{Y}) \longrightarrow 0$$

and there is a map $\mu: H^*(E, \overline{Y}) \longrightarrow H^*(\Omega C\times E, E)$ such that

$$j^*\circ\mu = m^* - l^*$$

where $m: \Omega C\times E \longrightarrow E$ is the action of the fibre on the total space and l is the obvious projection (see Thomas [36]).

We drop the subscript for ι_s . Then if $v\otimes a \in \mathcal{A}_2(Y)$

$(v\otimes a).(1\otimes\iota) = v\otimes a(\iota)$.

Then using an exact sequence of E.Thomas [36] we deduced that there exists a class $\phi \in H^*(E, Y)$ such that $\mu(\phi) = \Sigma\,\alpha_i.(1\otimes\iota_i)$

where by abuse of notation we have denoted the fundamental class of K^i in $H^*(C)$ by ι_i. If $s+q < 2$ (connectivity of C) then ϕ is determined modulo ($Ker(\mu) =$) $Im(p_1^*) = p^*_1 H^{s+q}(Y \times K_s, Y)$. Then $\bar{\Phi}$ the coset of ϕ modulo $p_1^*(H^{s+q}(Y \times K_s, Y)$ determines a family of operations associated with the relation $\alpha . \beta = 0$.

Choose ϕ to be a representative then $\Phi(u)$ is defined on cohomology class $u \in H^s(X, B)$ which satisfies

$$(\xi \times u\ d)^*(\beta.(1 \otimes \iota)) = 0 ;$$

and $\bar{\Phi}(u, \xi) = \{ f^*(\phi) \mid f:(X,B) \longrightarrow (E,Y)$ a lifting of $(\xi \times u) \circ d$ to $E\}$

where $d: X \longrightarrow X \times X$ is the diagonal map ; $\bar{\Phi}(u, \xi)$ is a coset modulo $Indet(X, B, \bar{\Phi}, \xi) = \Sigma_i \alpha_i . H^*(X, B)$ and is called a second order or secondary twisted cohomology operation.

A k^{th} order operation is defined analogously using the method of universal example. We suppose that

$$p_{k-1} : (E^{k-2}, Y^{k-2}) \longrightarrow (Y \times K_s, Y)$$

a universal example of $(k-1)^{th}$ order is defined ($k > 2$) .

Suppose there exist elements $\alpha_1, \ldots, \alpha_k \in \mathcal{O}_2(Y)$ and cohomology classes $\phi_1, \phi_2, \ldots, \phi_k \in H^*(E^{k-2}, Y^{k-2})$ such that

(1.3.3) $\quad \Sigma_i \alpha_i . \phi_i = 0$ is a relation of degree $t+1$

where $\mathcal{O}_2(Y)$ acts on $H^*(E^{k-2}, Y^{k-2})$ via

$$E^{k-2} \xrightarrow{p_{k-1}} Y \times K_s \xrightarrow{1} Y \quad .$$

Define E^{k-1} by using $(\phi_1, \phi_2, \ldots, \phi_k)$ as the classifying map. Then as before we get a class $\theta \in H^*(E^{k-1}, Y^{k-1})$ such that

$$\mu(\theta) = \Sigma_i \alpha_i . (\iota_i \otimes 1)$$

where $\bar{\iota}_i$ is the fundamental class of $K^i = K(Z_2, \deg\phi_i)$ in $C_{k-1} = K^1 \times K^2 \times \ldots \times K^k$ which is the classifying space for the principal bundle $(E^{k-1}, Y^{k-1}) \longrightarrow (C_{k-1}, *)$. If $t < 2$(connectivity of C_{k-1}) , then θ is determined modulo $\mathrm{Ker}(\mu) = \mathrm{Im}(p^*_{k-1})$. Thus the coset of θ with respect to $\mathrm{Im}(p^*_{k-1})$ defines a family of k^{th} order twisted cohomology operations $\bar{\Theta}$; $\Theta(u,\xi)$ is defined if $\Phi_{k-1}(u,\xi)$ is defined and $0 \in \Phi_{k-1}(u,\xi)$ where $\Phi_{k-1}(u,\xi)$ is vector valued i.e. $\Theta(u,\xi)$ is defined iff $0 \in (\phi_1, \phi_2, \ldots, \phi_k)(u,\xi)$. $\Theta(u,\xi)$ is defined to be

$$\Theta(u,\xi) = \cup \{ f^*_{k-1}(\theta) \mid f_{k-1} : (X,B) \longrightarrow (E^{k-1}, Y^{k-1}) \text{ a lifting of } (\xi \times u) \circ d \text{ to } E^{k-1} \} .$$

Its indeterminacy is defined inductively as follows:

The indeterminacy of $\Phi_{k-1}(u,\xi)$ is given by $\mathrm{Indet}^{t_1, t_2, \ldots, t_k}(X,B; \phi_1, \phi_2, \ldots, \phi_k; \xi)$

is defined to be the least subgroup of

$$H^{t_1}(X,B) \oplus H^{t_2}(X,B) \oplus \ldots \oplus H^{t_k}(X,B)$$

containing all classes of the form

$$f^*_{k-2}(\phi_1, \phi_2, \ldots, \phi_k) - f'^*_{k-2}(\phi_1, \ldots, \phi_k)$$

where $t_i = \deg\phi_i$, f_{k-2} and f'_{k-2} are arbitrary liftings of $(\xi \times u) \circ d$ to E^{k-2}. ($(\phi_1, \phi_2, \ldots, \phi_k)(u,\xi) = \bigcup_f (f^*(\phi_1), \ldots, f^*(\phi_k))$ where the union is taken over all liftings $f : (X,B) \longrightarrow (E^{k-2}, Y^{k-2})$ of $(\xi \times u) \circ d$ to E^{k-2}.) Since $p_{k-1} : (E^{k-1}; Y^{k-1}) \longrightarrow (E^{k-2}, Y^{k-2})$ is a principal bundle, two non-homotopic liftings differ by a map into its fibre. Inductively we see that the indeterminacy of Θ is given by $\Theta(0,\xi)$. For more details the reader is asked to consult E. Thomas [29].

1.3.3. The Generating Class Theorem.

Consider the following situation:

$$\begin{tikzcd} \Omega C_1 \arrow[r, "i_1"] & E^1 \arrow[r, "k"] \arrow[d, "p_1"] & K_t \\ B' \arrow[ur, dashed, "h_1"] \arrow[r, "\pi"'] & B \arrow[r, "k^1"] & C_1 = K_{i_1} \times \dots \times K_{i_a} \end{tikzcd}$$

where $p_1 : E^1 \longrightarrow B$ is a principal fibration induced by $k^1 ; B \longrightarrow C_1$. Let κ be the coset of k in $H^t(E^1)$ with respect to $\mathrm{Ker}\,(h_1^*) \cap \mathrm{Ker}(\mu) \cap H^t(E^1)$. (We have assumed that $\pi^*(k^1) = 0$ so that a lifting $h_1 : B' \longrightarrow E^1$ exists.) We suppose that

$$\mu(k) = \hat{\alpha} \cdot \iota \quad \text{and } h_1^*(k) = 0$$

where $\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_a)$, $\hat{\alpha}_i \in \mathfrak{A}_2(B)$, $\iota = (\iota_{i_1} \otimes 1, \dots, \iota_{i_a} \otimes 1) \in H^*(\Omega C_1 \times E^1)$ and $\mu : H^*(E^1) \longrightarrow H^*(C_1 \times E^1, E^1)$ is as defined in 1.3.2. The $\mathrm{Indet}^*(B' ; \kappa)$ is defined to be $\hat{\alpha} \cdot H^*(B')$.

Definition. A cohomology class $v \in H^s(B)$ is a generating class for κ if there exists a space Y, a map $\eta : B \longrightarrow Y$ and vectors $\alpha, \beta \in \mathfrak{A}_2(Y)$, operations $\phi_1, \phi_2, \dots, \phi_a, \psi_1, \dots, \psi_b$ defined over $\mathfrak{A}_2(Y)$ and a vector ρ in $\mathfrak{A}(2)$ such that

(1) $\hat{\alpha} = \eta^*(\alpha \times \rho)$ where $\eta^* : \mathfrak{A}_2(Y) \longrightarrow \mathfrak{A}_2(B)$;

(2) $(\rho \times k^1, 0) \in (\phi, \psi)v$ where $\phi = (\phi_1, \phi_2, \dots, \phi_a)$, $\psi = (\psi_1, \dots, \psi_b)$ and a relation of degree n say,

$$|\alpha| \cdot \phi + |\beta| \cdot \psi = 0 \qquad ; \text{ and}$$

there is an operation Ω associated with the above relation such that $\Omega(\pi^* v, \pi\eta) = \pi^* I$ where I is a coset of $\mathrm{Indet}^{s+n}(B, \Omega, \eta)$ in $H^{s+n}(B)$.

1.3.4. THEOREM. (E. Thomas) With notation as in 1.3.3, there exists a class $e \in \kappa$ such that

$$e \in \Omega(p_1^*(v),\ p_1^*(\eta)\ ) - p_1^*(M)$$

where v is the generating class for κ by hypothesis .

For the proof of this theorem known as the Generating Class Theorem we refer the reader to Thomas [29] .

1.3.5. Remark. Notice that

$$\mathrm{Indet}^t(X;\kappa) \subset \mathrm{Indet}^t(X,\Omega,\xi^*\eta) \quad \text{for any map } \xi : X \longrightarrow B.$$

If $\xi^*(k^1) = 0$ and $\mathrm{Indet}^t(X;\kappa) = \mathrm{Indet}^t(X,\Omega,\xi^*\eta)$ then

$$e(\xi) = \{\Omega(\xi^*(v),\ \xi^*(\eta))\} - \{\xi^*(M)\}$$

where $\{\ \ \}$ is a coset with respect to $\mathrm{Indet}^t(X;\kappa)$.

1.3.6. Definition. We call the operation Ω the <u>operation corresponding to</u> the <u>cohomology class $k \in H^t(E^1)$</u> or to κ . If k is a vector and if the generating class is the same for each component we shall drop the reference to the individual component.

1.3.7. Let M be a manifold of dimension n odd $\equiv 7$ (8) satisfying conditions (A) and (B) of 1.1.1. Let $\tau : M \longrightarrow BSpin_n$ be the classifying map for the tangent bundle of M. Then $H^*(M)$ is an $\mathcal{A}_2(BSpin_n)$-module; and the lifting $h_1 : M \longrightarrow E^1$ induces a morphism of $\mathcal{A}_2(BSpin_n)$-modules.

Recall the portion of the tower (1.2.1') :

$$(1.3.8)\qquad \begin{tikzcd} F^1 \arrow[r] & V_{n,7} \arrow[r] \arrow[d, "i_1"] & BSpin_{n-7} \arrow[d, "p_1"] & \\ & \Omega C_1 \arrow[r] & E^1 \arrow[r, "k^2"] \arrow[d, "q_1"] & C_2 \\ & & BSpin_n \arrow[r, "(\delta w_{n-7}, w_{n-5}, w_{n-3})"] & C_1 \end{tikzcd}$$

Let $\gamma = 1 \otimes Sq^4 + \iota_4 \otimes 1$ and $\delta = Sq^1 \iota_4 \otimes 1$ be in $\mathfrak{A}_2(K(Z_2,4))$. Then we have the following relation in $\mathfrak{A}_2(K(Z_2,4))$:

$$(1.3.9)\quad \gamma.\gamma + Sq^2(\gamma.Sq^2) + Sq^1(Sq^2\gamma.Sq^1) + \delta.(Sq^2Sq^1) = 0 .$$

1.3.10. CLAIM. w_{n-7} is a generating class for k^2_6.

Proof. This is due to Thomas essentially but we shall find the corresponding cohomology operation and show that it is not very different from that considered by E. Thomas in his preprint to [29]. In fact it is the secondary twisted cohomology operation associated to the relation (1.3.9).

Using Wu's formula we have that

$$Sq^2 w_{n-7} = w_{n-5},\ Sq^1 w_{n-7} = \delta w_{n-7} \bmod 2,\ Sq^2 Sq^1 w_{n-7} = 0,$$
$$\gamma.Sq^2 w_{n-7} = 0,\ \gamma.Sq^1{}_{n-7} = 0,\ \gamma.w_{n-7} = w_{n-3}.$$

So we let $\alpha = (0,0,1\otimes Sq^4 + w_4 \otimes 1)$ a vector in $\mathfrak{A}_2(BSpin_n)^3$. We shall find cohomology operations (twisted) ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2, ψ_3 and a vector β in $\mathfrak{A}_2(K_4)$ such that

(1) $|\alpha|.\phi + |\beta|.\psi = 0$,

(2) $(\phi,\psi).w_{n-7} = (\rho \times k^1, 0)$ where $k^1 = (\delta w_{n-7}, w_{n-5}, w_{n-3})$.

We take $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \gamma = (1\otimes Sq^4 + \iota \otimes 1)$, $\eta : BSpin_n \longrightarrow K_4$ be w_4, $\rho = (0,0,1)$, $\psi = (\gamma.Sq^2, Sq^2\gamma.Sq^1, Sq^2Sq^1) = (Sq^2, Sq^1, \delta)$, and $\phi = (0, 0, \gamma)$.

$$
(1.3.8)\qquad
\begin{tikzcd}
F^1 \arrow[r] & V_{n,7} \arrow[r] \arrow[d] & BSpin_{n-7} \arrow[d, "p_1"'] & & \\
& \Omega C_1 \arrow[r, "i_1"] & E^1 \arrow[r, "k^2"] \arrow[d, "q_1"'] & C_2 & \\
& & BSpin_n \arrow[rr, "(\delta w_{n-7}, w_{n-5}, w_{n-3})"] & & C_1
\end{tikzcd}
$$

Let $\gamma = 1\otimes Sq^4 + \iota_4\otimes 1$ and $\delta = Sq^1\iota_4\otimes 1$ be in $\mathfrak{A}_2(K(Z_2,4))$. Then we have the following relation in $\mathfrak{A}_2(K(Z_2,4))$:

(1.3.9) $\gamma.\gamma + Sq^2(\gamma.Sq^2) + Sq^1(Sq^2\gamma.Sq^1) + \delta.(Sq^2Sq^1) = 0$.

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Proof. This is due to Thomas essentially but we shall find the corresponding cohomology operation and show that it is not very different from that considered by E. Thomas in his preprint to [29]. In fact it is the secondary twisted cohomology operation associated to the relation (1.3.9).

Using Wu's formula we have that

$$Sq^2 w_{n-7} = w_{n-5},\ Sq^1 w_{n-7} = \delta w_{n-7} \bmod 2\ ,\ Sq^2Sq^1 w_{n-7} = 0\ ,$$
$$\gamma.Sq^2 w_{n-7} = 0\ ,\ \gamma.Sq^1{}_{n-7} = 0\ ,\ \gamma.w_{n-7} = w_{n-3}.$$

So we let $\hat{\alpha} = (0,0,1\otimes Sq^4 + w_4\otimes 1)$ a vector in $\mathfrak{A}_2(BSpin_n)^3$. We shall find cohomology operations (twisted) ϕ_1, ϕ_2, ϕ_3 , ψ_1, ψ_2, ψ_3 and a vector β in $\mathfrak{A}_3(K_4)$ such that

(1) $|\alpha|.\phi + |\beta|.\psi = 0$,

(2) $(\phi,\psi).w_{n-7} = (\rho\times k^1, 0)$ where $k^1 = (\delta w_{n-7}, w_{n-5}, w_{n-3})$.

We take $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \gamma = (1\otimes Sq^4 + \iota\otimes 1)$, $\eta : BSpin_n \longrightarrow K_4$ be w_4 , $\rho = (0,0,1)$, $\psi = (\gamma.Sq^2, Sq^2\gamma.Sq^1, Sq^2Sq^1) = (Sq^2, Sq^1, \delta)$, and $\phi = (0, 0, \gamma)$.

Let Ω be a secondary twisted cohomology operation associated with (1.3.9). Since $\pi: BSpin_{n-7} \longrightarrow BSpin_n$ induces epimorphism in cohomology $\Omega(\pi^* w_{n-7}, \pi^* w_4) = \pi^* N$ where N is a coset of the indeterminacy $\mathrm{Indet}^n(BSpin_n, \Omega, w_4)$. Thus the claim is justified.

1.3.11. We shall evaluate this operation by the method of universal example. We set the following notation:

$\alpha = (\gamma, Sq^2, Sq^1, \delta)$, $= (\gamma, \gamma . Sq^2, Sq^2\gamma Sq^1, Sq^2 Sq^1)$;

$\beta_1 : (K_4 \times K_{n-7}, K_4 \times *) \longrightarrow K_{n-3}$,

$\beta_2 : (K_4 \times K_{n-7}, K_4 \times *) \longrightarrow K_{n-1}$,

$\beta_3 : (K_4 \times K_{n-7}, K_4 \times *) \longrightarrow K_n$ and

$\beta_4 : (K_4 \times K_{n-7}, K_4 \times *) \longrightarrow K_{n-4}$

representing respectively $\gamma.(1 \otimes \iota_{n-7})$, $(\gamma Sq^2).(1 \otimes \iota_{n-7})$, $(Sq^2 \gamma Sq^1).(1 \otimes \iota_{n-7})$ and $(Sq^2 Sq^1).(1 \otimes \iota_{n-7})$;

set $C = K_{n-3} \times K_{n-1} \times K_n \times K_{n-4}$.

We have the following tower of universal example for Ω:

$$\begin{tikzcd}
 & \Omega C \arrow[r] & E^1 \arrow[r] \arrow[d, "p"'] & K_n \\
BSpin_{n-7} \arrow[rr] \arrow[urr, dashed] & & K_4 \times K_{n-7} \arrow[r, "\beta.(1 \otimes \iota_{n-7})"] & C
\end{tikzcd} .$$

Consider the fibre square:

$$(1.3.12) \qquad \begin{tikzcd}
\Omega C \times E^1 \arrow[r, "m"] \arrow[d, "proj"'] & E^1 \arrow[d, "p"] \\
E^1 \arrow[r, "p"] & K_4 \times K_{n-7}
\end{tikzcd} .$$

which defines the action of the fibre on the total space E^1, $m: \Omega C \times E^1 \longrightarrow E^1$. Using this fibre square, Thomas derived the following long exact sequence :

$$(1.3.13)\quad \ldots\ H^{i}(\Omega C\times E^1, E^1)\xrightarrow{\tau} H^{i+1}(K_4\times K_{n-7})\xrightarrow{p^*} H^{i+1}(E^1)\xrightarrow{\mu}$$

$$\longrightarrow H^{i+1}(\Omega C\times E^1, E^1)\xrightarrow{\tau}\ \ldots \longrightarrow H^{2(n-5)-1}(K_4\times K_{n-7})\xrightarrow{p^*}$$

$$\longrightarrow H^{2(n-5)-1}(E^1)\xrightarrow{\mu} H^{2(n-5)-1}(\Omega C\times E^1, E^1)\xrightarrow{\tau} H^{2(n-5)}(K_4\times K_{n-7})$$

$$\xrightarrow{p^*} H^{2(n-5)}(E^1)$$

where $\mu : H^*(E^1)\longrightarrow H^*(\Omega C\times E^1, E^1)$ is as defined in 1.3.2 when absolute cohomology is considered and $\tau : H^{i}(\Omega C\times E^1, E^1)\longrightarrow H^{i+1}(K_4\times K_{n-7})$ for $i \leqslant 2(n-5)$ is given by the composite:

$$H^{i}(\Omega C\times E^1, E^1)\xrightarrow{j^*} H^{i}(\Omega C\times E^1)\xrightarrow{\tau_1} H^{i+1}(K_4\times K_{n-7})$$

where $\tau_1 : H^{i}(\Omega C\times E^1)\longrightarrow H^{i+1}(K_4\times K_{n-7})$ is as defined in [28]. According to Thomas [36] τ is an $\mathcal{O}_2(K_4)$-morphism where $\mathcal{O}_2(K_4)$-structure on $\Omega C\times E^1$ is given by the map $\Omega C\times E^1\to E^1\to K_4\times K_{n-7}\to K_4$. Therefore, since $\tau(\iota_{n-4})=\beta_1$, $\tau(\iota_{n-2})=\beta_2$, $\tau(\iota_{n-1})=\beta_3$ and $\tau(\iota_{n-5})=\beta_4$,

$$\tau\{\alpha_1.(\iota_{n-4}\otimes 1)+\alpha_2.(\iota_{n-2}\otimes 1)+\alpha_3.(\iota_{n-1}\otimes 1)+\alpha_4.(\iota_{n-5}\otimes 1)\}$$

$$= (\ \Sigma\alpha_i.\beta_i\ ).(1\otimes\iota_{n-7}) = 0 .$$

By exactness of 1.3.13 there exists $\theta\in H^{n}(E^1)$ such that

$$\mu(\theta) = \alpha_1.(\iota_{n-4}\otimes 1)+\alpha_2.(\iota_{n-2}\otimes 1)+\alpha_3.(\iota_{n-1}\otimes 1)+\alpha_4.(\iota_{n-5}\otimes 1).$$

It is obvious that Ω is defined on $(\pi^* w_{n-7}, \pi^* w_4)$ and $\Omega(\pi^* w_{n-7}, \pi^* w_4)$ is a coset with respect to $\mathrm{Indet}^n(BSpin_{n-7}, \Omega, w_4)$

$$= \gamma.H^{n-4}(BSpin_{n-7})\oplus Sq^2H^{n-2}(BSpin_{n-7})\oplus Sq^1H^{n-1}(BSpin_{n-7})$$
$$\oplus\,\delta.H^{n-5}(BSpin_{n-7})$$

$$= (Sq^4+w_4.)H^{n-4}(BSpin_{n-7})\oplus Sq^2H^{n-2}(BSpin_{n-7})\oplus Sq^1H^{n-1}(BSpin_{n-7})$$

since $\delta.H^{n-5}(BSpin_{n-7}) = (Sq^1 w_4).H^{n-5}(BSpin_{n-7}) = 0$.

Since π^* is an epimorphism, $(\pi^* w_{n-7}, \pi^* w_4) = \pi^* N$ where N is a coset with respect to $\mathrm{Indet}^n(BSpin_n, \Omega, w_4)$. Therefore by Theorem 1.3.3 we have the following:

1.3.14. LEMMA. Notation as of 1.3.7. There exists a class $e \in H^n(E^1)$ in the coset of the k-invariant k_6^2 with respect to $\mathrm{Ker}(p_1^*) \cap \mathrm{Im}(q_1^*)$ such that

$$e \in (q_1^*(w_{n-7}), q_1^*(w_4)) - q_1^*(N)$$

where E^1 is the first stage of the Postnikov tower (1.2.14) and N is given by the paragraph preceding this.

1.3.15. Suppose $\tau : M \longrightarrow BSpin_n$ is the classifying map for the tangent bundle of M. By the connectivity condition on M we see that $\mathrm{Indet}^n(M, \kappa) = 0 = \mathrm{Indet}^n(M, \Omega, \tau^*(w_4))$ where κ is the coset of k_6^2 with respect to $\mathrm{Ker}(p_1^*) \cap \mathrm{Im}(q_1^*)$. Then by naturality, Lemma 1.3.14 and the fact that $\mathrm{Ker}(p_1^*) \cap \mathrm{Im}(q_1^*) = 0$, we have

$$k_6^2(\tau) = \Omega(\tau^*(w_{n-7}), \tau^*(w_4)) - \{\tau^*(N)\} \qquad (1.3.16)$$

with zero indeterminacy.

By the connectivity condition

$$(1.3.17) \quad \Omega(\tau^*(w_{n-7}), \tau^*(w_4)) = \Omega(\tau^*(w_{n-7}), 0)$$

and so Ω on $\tau^*(w_{n-7})$ reduces to an Adams-Maunder operation [[illegible]] associated to (1.3.9). Thus τ lifts to E^2 of (1.3.19) iff $0 \in k_6^2(\tau)$ iff $0 = \Omega(\tau^*(w_{n-7}), \tau^*(w_4)) - \tau^*(N)$

$$\underline{\text{or}}\ \Omega(\tau^*(w_{n-7}), \tau^*(w_4)) = \tau^*(N).$$

1.3.18. LEMMA. Let U_n be the universal Thom class over $BSpin_n$. Then Ω is defined on U_n and $0 \in \Omega(U_n, w_4)$.

Proof. It is easy to see that Ω is defined on U_n where the twisting is via the universal disc bundle by $w_4 \in H^4(BSpin_n)$. Now, $H^{n+7}(TBSpin_n) \cong Z_2 = \{U_n.w_7\} = \{Sq^1(U_n.w_6)\} \in Indet^{n+7}(TBSpin_n, \Omega, w_4)$, therefore by Thomas-Peterson [35], $0 \in \Omega(U_n, w_4)$. Here we have denoted the Thom complex of the universal n-plane bundle over $BSpin_n$ by $TBSpin_n$.

1.3.19. LEMMA. (E. Thomas).

$$0 \in \Omega(\pi^*(w_{n-7}), \pi^*(w_4)) \subset H^n(BSpin_{n-7}).$$

Proof. By Lemma 1.3.18 $0 \in \Omega(U_{n-7}, w_4)$. Since $s^*(U_{n-7}) = w_{n-7}$ mod 2 where $s: BSpin_{n-7} \longrightarrow TBSpin_{n-7}$ is the 'zero section', the assertion follows from the naturality of twisted cohomology operations.

Since $Indet^n(BSpin_{n-7}, \Omega, w_4) \cap Im(s^*) = s^*(Indet^n(TBSpin_{n-7}, \Omega, w_4))$, we can take N to be $Indet^n(BSpin_n, \Omega, w_4)$ by 1.3.19.

Thus we have:

1.3.20. LEMMA. τ lifts to E^2 iff $0 \in \Omega(\tau^*(w_{n-7}), \tau^*(w_4))$.

1.3.21. THEOREM. $0 = \Omega(\tau^*(w_{n-7}), 0)$. Hence τ lifts to E^2.

Proof. Without twisting the operation associated with (1.3.9) is just the ordinary secondary operation, $\tilde{\Omega}$, associated with the relation:

$$(1.3.22) \quad Sq^4Sq^4 + Sq^2(Sq^4Sq^2) + Sq^1(Sq^2Sq^4Sq^1) = 0 \quad .$$

In general a secondary cohomology operation associated with a relation in $\mathcal{A}(2)$ say,

$$\Sigma_i \, \alpha_i \beta_i = 0 \quad , \qquad \alpha_i, \beta_i \in \mathcal{A}(2) \; ,$$

is defined analogously using the method of universal example as 1.3.2. For a detail exposition we refer the reader to Adams [1] and for a generalization to higher order cohomology operations we refer to Maunder [21] .

However, we shall consider Adams' basic operation associated with the relation:

$$(1.3.23) \quad Sq^4 Sq^4 + (Sq^2 Sq^4) Sq^2 + (Sq^1 Sq^2 Sq^4) Sq^1 = 0 \; .$$

We use the standard notation $\Phi_{2,2}$ for this operation. We want to compare this operation with $\widetilde{\Omega}$. Since $w_1(M) = w_2(M) = w_3(M) = w_4(M) = w_5(M) = w_6(M) = w_7(M) = 0$, both $\widetilde{\Omega}(w_{n-7}(M))$ and $\Phi_{2,2}(w_{n-7}(M))$ have zero indeterminacy by Wu duality.

Following Adams we explore the relation between $\widetilde{\Omega}$ and $\Phi_{2,2}$. First we show that on $w_{n-7}(M)$ the operations are the same.

1.3.24. LEMMA. $\widetilde{\Omega}(w_{n-7}(\tau)) = \Phi_{2,2}(w_{n-7}(\tau))$.

Proof. Let

$$\hat{\alpha} = (Sq^4, Sq^2, Sq^1) \; , \; \hat{\beta} = (Sq^4, Sq^4 Sq^2, Sq^2 Sq^4 Sq^1)$$

$$\alpha = (Sq^4, Sq^2 Sq^4, Sq^1 Sq^2 Sq^4) \; , \; \beta = (Sq^4, Sq^2, Sq^1) \quad .$$

We note that $\hat{\beta} = (1, Sq^4, Sq^2 Sq^4) \times \beta$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\Omega C_1 \arrow[r, "i"] \arrow[d, "\bar{h}"] & E_1 \arrow[r, "p"] \arrow[d, "f"] & K_{n-7} \arrow[r, "\beta"] \arrow[d, equal] & K_{n-3}\times K_{n-5}\times K_{n-6} = C_1 \arrow[d, "h"', "\{1,\, Sq^4,\, Sq^2Sq^4\}"] \\
\Omega \hat{C}_1 \arrow[r, "\hat{i}"] & \hat{E}_1 \arrow[r, "\hat{p}"] & K_{n-7} \arrow[r, "\hat{\beta}"] & K_{n-3}\times K_{n-1}\times K_n = C_1
\end{tikzcd}$$

Then according to Thomas [36], we have the following commutative diagram with the horizontal rows exact:

$$\begin{tikzcd}
\longrightarrow H^i(K_{n-7}) \arrow[r, "p^*"] & H^i(E_1) \arrow[r, "\mu"] & H^i(\Omega C_1\times E_1, E_1) \arrow[r, "\tau"] & H^{i+1}(K_{n-7}) \\
\longrightarrow H^i(K_{n-7}) \arrow[r, "\hat{p}^*"] \arrow[u, equal] & H^i(\hat{E}_1) \arrow[r, "\hat{\mu}"] \arrow[u, "f^*"] & H^i(\Omega\hat{C}_1\times\hat{E}_1, \hat{E}_1) \arrow[r, "\tau"] \arrow[u, "(\bar{h}\times f)^*"] & H^{i+1}(K_{n-7}) \arrow[u, equal]
\end{tikzcd}$$

for $i \leq 2(n-7)-1$.

Let $\hat{\theta} \in H^n(\hat{E}_1)$ be a representative for $\tilde{\Omega}$; and let $\theta \in H^n(E_1)$ be a representative for $\bar{\Phi}_{2,2}$. Then using the above diagram we have

$$\mu(f^*(\hat{\theta}) - \theta) = (h\times f)^*\hat{\mu}(\hat{\theta}) - \mu(\theta)$$

$$= 0 \quad \text{by definition of } h \text{ .}$$

Therefore $f^*(\hat{\theta}) = \theta$ modulo $\mathrm{Im}(p^*)$. This implies that

$$(\hat{\theta}\circ f)\circ\zeta - \theta\circ\zeta \equiv 0 \text{ modulo total indeterminacies}$$

where ζ is a lifting of $w_{n-7}(\tau) : M \longrightarrow K_{n-7}$ to E_1 (since $i^*(f^*(\hat{\theta})-\theta) = 0$ in $H^n(\Omega C_1)$). Thus $\tilde{\Omega}(w_{n-7}(\tau)) = \bar{\Phi}_{2,2}(w_{n-7}(\tau))$. This completes the proof of Lemma 1.3.24.

We now return to the proof of Theorem 1.3.21. Thomas in his preprint to [29] showed that $0 \in \bar{\Phi}_{2,2}(w_{n-7}(\tau))$. Essentially the (S-) dual operation $\chi\bar{\Phi}_{2,2}$ of $\bar{\Phi}_{2,2}$ (See Maunder [31] for a definition of dual operation .) is always zero on the Thom class of the universal N-plane bundle over

$\widehat{BSpin}_N$ which classifies any spin N-plane bundle γ satisfying $w_4(\gamma) = 0$, for $N > 7$ since $H^{N+7}(T\widehat{BSpin}_N) = Sq^1H^6(T\widehat{BSpin}_N) \subset \mathrm{Indet}^{N+7}(T\widehat{BSpin}_N, \chi\Phi_{2,2})$ (because a spectral sequence argument shows that $H^7(\widehat{BSpin}_N) = Sq^1H^6(\widehat{BSpin}_N)$). Now the Thom complex $T\nu$ of the normal bundle of an embedding $\nu: M \subset R^{n+N}$ for some sufficiently large N is the S-dual of $M \cup$ a point. Since ν also satisfies $w_4(\nu) = w_2(\nu) = w_1(\nu) = 0$ ν is classified by a map from M to $\widehat{BSpin}_N$ also denoted by ν. Let U_ν be the Thom class of the normal bundle ν. Then by naturality

$$\chi\Phi_{2,2}(U_\nu) = \{0\} .$$

Alexander-Pontryagin duality then implies that $\Phi_{2,2}$ is trivial on the domain of definition of $\Phi_{2,2}$ of dimension n-7 if and only if the dual operation $\chi\Phi_{2,2}$ is trivial on the Thom class U_ν. (See Maunder [21] for this assertion.) Thus $\Phi_{2,2}(w_{n-7}(\tau)) = 0$ modulo zero indeterminacy. Hence we have completed the proof of Theorem 1.3.21.

We now come to the last obstruction in the third stage. We would like to have a 'generating class theorem' to relate the final obstruction to a tertiary twisted cohomology operation. However, the situation is not satisfactory for it is not hard to observe from the defining relation of the k-invariants for the second stage of the n-MPT (1.2.19) that there is no natural candidate for a generating class for k_i^2, $i = 1, \ldots, 6$ simultaneously. This forces us to look at the 'corresponding operations' on the Thom class. This will be the main programme of the next two chapters.

CHAPTER 2. ADMISSIBLE CLASS THEOREM

In this chapter we shall identify the k-invariants for the second stage of an n-MPT for the fibration,

$$V_{n,7} \longrightarrow BSpin_{n-7} \longrightarrow BSpin_n \text{ for } n \equiv 7 \ (8) > 15,$$

via the Thom isomorphism to some secondary cohomology operations on the Thom class of the bundle over E^1, the first stage of the n-MPT, induced from the universal spin n-plane bundle over $BSpin_n$.

§2.1. Admissible Class

2.1.1. Definition. Suppose we have the following situation:

$$\begin{tikzcd}
\Omega C \arrow[r, "i"] & E \arrow[r, "\theta"] \arrow[d, "p"] & K_t \\
A \arrow[ur, dashed, "q"] \arrow[r, "\pi"'] & B \arrow[r, "k"] & C
\end{tikzcd}$$

where C is a product of K_q or K_s^*, $t < 2j$, j = connectivity of C, A and B are spaces and $p: E \longrightarrow B$ is the principal fibration induced by the vector of cohomology classes $k: B \longrightarrow C$. Suppose $\theta \in H^t(E)$ is a class such that

$$q^*(\theta) = 0, \quad \mu(\theta) = \alpha \cdot \iota, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_a)$$

where $\alpha_i \in \mathcal{A}(2)$ for $i = 1, 2, \ldots, a$, and

$\iota = (\iota_1, \iota_2, \ldots, \iota_a)$ where $\{\iota_i\}_{i \leq a}$ are the respective fundamental classes of the Eilenberg-MacLane spaces in C. Then the class k or vector k is said to be <u>admissible</u> if there exist a q-plane bundle, γ, over B, operations (possibly higher order i.e. n^{th} order $n > 2$) $\phi_1, \phi_2, \ldots, \phi_a, \psi_1, \psi_2, \ldots, \psi_b$, and vectors $\alpha \in \mathfrak{A}(2)^a$ and $\beta \in \mathfrak{A}(2)^b$ such that

(1) $(\phi, \psi)(U_\gamma) = (\Psi_B(k), 0)$ where U_γ is the Thom class of γ, Ψ_B is the Thom isomorphism in cohomology, $\phi = (\phi_1, \phi_2, \ldots, \phi_a)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_b)$;

(2) $|\alpha|.\phi + |\beta|.\psi = 0$ is a relation of degree t+1 on integral cohomology classes; and

(3) there exists an operation Ω associated with the relation in (2) above such that

$$\Omega(T\pi^*(U_\gamma)) = T\pi^*(N)$$

where $N \subset H^{q+t}(TB)$ is a coset with respect to $\mathrm{Indet}^{t+q}(TB, \Omega)$ and where T is the functor that assigns to every space over B the Thom complex of the bundle induced from the q-plane bundle γ over B.

Note that TB is just $T\gamma$ the Thom complex of γ.

We shall be interested in the fibration,

(2.1.2) $\quad V_{n,7} \longrightarrow BSpin_{n-7} \longrightarrow BSpin_n \quad$ for $n \equiv 7(8) > 15$,

only; so from now on n will be an integer $\equiv 7(8) > 15$.

2.1.3. Definition. Assume $\pi^*(k) = 0$ in2.1.1. (This assumption is implicit in 2.1.1.) Let $q: A \longrightarrow E$ be a lifting of π to E.and let Θ be the coset of θ with respect to

$$\operatorname{Ker}(q^*) \cap \operatorname{Im}(p^*) \cap H^t(E) \quad .$$

We say the class k is <u>admissible</u> <u>with</u> <u>respect to θ</u> or <u>Θ</u>; and we refer the cohomology operation associated with 2.1.1(2) as the <u>cohomology</u> <u>operation</u> <u>corresponding</u> <u>to</u> <u>θ</u> . If θ is a vector and k is admissible with respect to each of the component then we shall drop the reference to the individual component.

Apply the functor T using the universal spin n-plane bundle over $\mathbf{BSpin}_n$ to the category of spaces over $\mathbf{BSpin}_n$. Thus from (1.3.8) we get

$$(2.1.4) \qquad \begin{tikzcd} TF' \arrow[r] & TV_{n,7} \arrow[r] \arrow[d] & \mathbf{TBSpin}_{n-7} \arrow[d, "Tp_1"] \\ & T\Omega C_1 \arrow[r, dash, "Ti_1"] & TE^1 \arrow[d, "Tq_1"] \\ & & \mathbf{TBSpin}_n \end{tikzcd}$$

By the usual sort of argument we see that the first k-invariants k^1 of an n-MPT for (2.1.2) is admissible and the corresponding cohomology operations are tabulated below giving the defining relations:

TABLE 2.1.5. 'Corresponding operation' for k^2

k-invariant	Corr. coho. op	Defining relation	Deg.
k_1^2	Φ_1^*	$Sq^2(\delta Sq^{n-7}) = 0$	n-5
k_2^2	Φ_2^*	$Sq^2(Sq^{n-5}) = 0$	n-4
k_3^2	Φ_3^*	$Sq^4(\delta Sq^{n-7}) + Sq^{n-4}Sq^2 = 0$	n-3
k_4^2	Φ_4^*	$(Sq^2Sq^1)Sq^{n-5} + Sq^1Sq^{n-3} = 0$	n-3
k_5^2	Φ_5^*	$Sq^4Sq^{n-5} + Sq^{n-3}Sq^2 = 0$	n-2
k_6^2	Φ_6^*	$Sq^4Sq^{n-3} + Sq^{n-1}Sq^2 = 0$	n

Hence we have proved

2.1.6. PROPOSITION. k^1 of (1.2.19) is admissible for k^2 of (1.2.19) where k^1, k^2 are the k-invariants for the 1st and 2nd stages of the n-MPT for (2.1.2) .

§ 2.2. An Admissible Class Theorem

We shall work towards an admissible class theorem along a series of lemmas which might prove to be useful later on. First we introduce some notation for convenience.

2.2.1. Definition. Let (ξ, $p: E \longrightarrow B$) be a q-plane bundle over B with base point * . Define the reduced Thom complex of ξ to be $T(\xi)/T(\xi\,|*)$.

Let B be a fixed space with base point * . Let γ be a fixed q-plane bundle over B. Define a functor $\tilde{T}$ from the category of pointed spaces over B to the category of pointed spaces as follows:

$\tilde{T}$ assigns to every based space X over B the reduced Thom complex of the q-plane bundle over B induced from γ, and to every based map between spaces over B the obvious map between the reduced Thom complexes.

If the base point is non-degenerate in the sense of Puppe then the reduced Thom complex behaves like the unreduced Thom complex except that it does not have a Thom class:

(1) If A is a based space over B then there is an isomorphism similar to the Thom isomorphism,

$$\tilde{\Psi}_A : H^i(A) \longrightarrow H^{q+i}(\tilde{T}A) \qquad \text{for } i > 0$$

in reduced cohomology with coefficients in Z_2; if γ is an oriented q-plane bundle then the isomorphism holds for any reduced cohomology with coefficients in an associative ring with unit 1;

(2) the action of the Steenrod algebra on $H^{q+i}(\tilde{T}A)$ is the same as for the group $H^{q+i}(TA)$ for $i > 0$ and

(3) if ζ over A is trivial then the reduced Thom complex is the q^{th} iterated suspension of A, $S^q(A)$ and the isomorphisms in (1) above is just the q^{th} iterated suspension homomorphisms.

For the details of these properties we refer the reader to Thomas [36A] .

2.2.2 . Consider the following diagram obtained by applying $\tilde{T}$ with $B = BSpin_n$ and γ the universal spin q-plane bundle over $BSpin_n$ to diagram (1.3.8) and putting together the fact that $\tilde{T}q_1$ lifts to $\hat{E}^1$ the total space of the principal bundle over $TBSpin_n$ induced by $\tilde{\Psi}_{BSpin_n}(k^1)$:

$$
(2.2.3)\qquad
\begin{tikzcd}
 & S^n(\Omega C_1)\approx\tilde{T}\Omega C_1 \arrow[r, "q"] \arrow[d, "\tilde{T}i_1"'] & \Omega L_1 \arrow[d, "\hat{i}_1"] & \\
 & \tilde{T}E^1 \arrow[r, "f"] \arrow[d, "\tilde{T}q_1"'] & \hat{E}^1 \arrow[dl, "\hat{p}_1"] & \\
\tilde{T}BSpin_{n-7} \arrow[ur, "\tilde{T}p_1"] \arrow[r, "\tilde{T}\pi"] & \tilde{T}BSpin_n \arrow[rr, "\tilde{\Psi}(k^1)"] & & K^*_{2n-6}\times K_{2n-5}\times K_{2n-3} = L_1
\end{tikzcd}
$$

where $f: \tilde{T}E^1 \longrightarrow \hat{E}^1$ is a lifting of $\tilde{T}q_1$ to $\hat{E}^1$ and $q: \tilde{T}\Omega C_1 \longrightarrow L_1$ is the restriction of f to $\tilde{T}\Omega C_1$.

We set up the following notation to indicate the obvious maps:

$\bar{f} : (\tilde{T}E^1, \tilde{T}\Omega C_1) \longrightarrow (\hat{E}^1, \Omega L_1)$,

$\hat{p}_{10} : (\hat{E}^1, \Omega L_1) \longrightarrow (\tilde{T}BSpin_n, *)$,

$q_{10} : (E^1, \Omega C_1) \longrightarrow (BSpin_n, *)$ and we set

$B = BSpin_n$. We shall prove the next lemma which is also true in a more general setting. The aim is to explain how it is done in general with-out introducing more technicalities.

2.2.4. LEMMA. With notation as given by 2.2.2, the lifing of $\tilde{T}q_1 : \tilde{T}E^1 \longrightarrow \tilde{T}B$ to $\hat{E}^1$,

$$f: \tilde{T}E^1 \longrightarrow \hat{E}^1 ,$$

can be chosen such that the restriction of f to $\tilde{T}\Omega C_1$,

$q: \tilde{T}\Omega C_1 \longrightarrow \Omega L_1$, satisfies

$$q^*(\iota^*_{2n-7}) = \Sigma^n(\iota^*_{n-7}) \ ,$$

$$q^*(\iota_{2n-6}) = \Sigma^n(\iota_{n-6}) \text{ and}$$

$$q^*(\iota_{2n-4}) = \Sigma^n(\iota_{n-4})$$ where Σ denotes the suspension homomorphism.

Proof. Let $\tau: H^*(\Omega L_1) \longrightarrow H^*(\tilde{T}B)$ in the range of dimensions where it is defined be the transgression in the fibration,

$$\Omega L_1 \longrightarrow \hat{E}^1 \longrightarrow \tilde{T}B \ .$$

Then by construction,

$$\tau(\iota^*_{2n-7}) = \tilde{\Psi}_B(k^1_1) \ , \ \tau(\iota_{2n-6}) = \tilde{\Psi}_B(k^1_2) \text{ and } \tau(\iota_{2n-4}) = \tilde{\Psi}_B(k^1_3).$$

Therefore,

$$\delta(\iota^*_{2n-7}) = \hat{p}^*_{10}\tilde{\Psi}_B(k^1_1); \ \delta(\iota_{2n-6}) = \hat{p}^*_{10}\tilde{\Psi}_B(k^1_2) \text{ and}$$

$\delta(\iota_{2n-4}) = \hat{p}^*_{10}\tilde{\Psi}_B(k^1_3)$ where $\delta: H^*(\Omega L_1) \longrightarrow H^{*+1}(\hat{E}^1, \Omega L_1)$ is the coboundary homomorphism. Thus

$$\bar{F}^* \hat{p}^*_{10}\tilde{\Psi}_B(k^1_1) = \bar{F}^*\delta(\iota^*_{2n-7}) = \delta q^*(\iota^*_{2n-7}) \ . \quad \text{But}$$

$$\bar{F}^* \hat{p}^*_{10}\tilde{\Psi}_B(k^1_1) = (\tilde{T}q_{10})^*\tilde{\Psi}_B(k^1_1) = \tilde{\Psi}_{E^1,\Omega C_1} \circ q^*_{10}(k^1_1)$$

$$= \tilde{\Psi}_{E^1,\Omega C_1}\delta(\iota^*_{n-7}) = \delta\tilde{\Psi}_{\Omega C_1}(\iota^*_{n-7}) \ .$$

Hence $$\delta q^*(\iota^*_{2n-7}) = \delta\tilde{\Psi}_{\Omega C_1}(\iota^*_{n-7}) \ .$$

Consider the following portion of the long exact sequence for the pair $(E^1, \Omega C_1)$:

$$(3.2.5) \quad H^{n-7}(E^1,\Omega C_1) \longrightarrow H^{n-7}(E^1) \rightarrow \underset{\underset{Z_2}{\wr\|}}{H^{n-7}(\Omega C_1)} \longrightarrow H^{n-6}(E^1,\Omega C_1)$$

Since $H^{n-7}(\Omega C_1) \cong Z_2$ and $\delta(\iota^*_{n-7}) = q^*_{10}(k^1_1) \neq 0$, δ is monomorphic in dimension n-7 . This implies via the reduced Thom isomorphism that $\delta: H^{2n-7}(\tilde{T}\Omega C_1) \longrightarrow H^{2n-6}(\tilde{T}\tilde{E}^1, \tilde{T}\Omega C_1)$ is monomorphic and so

$$q^*(\iota^*_{2n-7}) = \tilde{\Psi}_{\Omega C_1}(\iota^*_{n-7}) = \Sigma^n(\iota^*_{n-7}).$$

Similarly, we can show that

$$q^*(\iota_{2n-6}) = \Sigma^n(\iota_{n-6}).$$

To prove that $q^*(\iota_{2n-4}) = \Sigma^n(\iota_{n-4})$ requires more work . Consider the following commutative diagram with exact rows:

$$(2.2.6)\qquad \begin{tikzcd} H^{2n-4}(\hat{E}^1) \arrow[r] \arrow[d, "f^*"] & H^{2n-4}(\Omega L_1) \arrow[r, "\delta"] \arrow[d, "q^*"] & H^{2n-3}(\hat{E}^1, \Omega L_1) \arrow[d, "\bar{f}^*"] \\ H^{2n-4}(\tilde{T}\tilde{E}^1) \arrow[r] & H^{2n-4}(\tilde{T}\Omega C_1) \arrow[r, "\delta"] & H^{2n-3}(\tilde{T}\tilde{E}^1, \tilde{T}\Omega C_1) \end{tikzcd} .$$

Now $H^{n-4}(\Omega C_1) \cong Z_2 \oplus Z_2 \oplus Z_2 \cong \langle Sq^3\iota^*_{n-7}, Sq^2\iota_{n-6}, \iota_{n-4} \rangle$ and so the kernel of $\delta: H^{n-4}(\Omega C_1) \longrightarrow H^{n-3}(E^1, \Omega C_1)$ is generated by $Sq^3\iota_{n-7}$ and $Sq^2\iota_{n-6}$ since

$$\delta(Sq^3\iota^*_{n-7}) = p^*_{10}(Sq^3k^1_1) = 0 \quad \text{and}$$

$$\delta(Sq^2\iota_{n-6}) = p^*_{10}(Sq^2k^1_2) = 0 .$$

As in the above cases $\delta q^*(\iota_{2n-4}) = \delta(\Sigma^n(\iota_{n-4}))$. By the commutativity and the exactness of the lower row of (2.2.6),

$$q^*(\iota_{2n-4}) = \Sigma^n(\iota_{n-4}) + a_1\Sigma^n(Sq^3\iota^*_{n-7}) + a_2\Sigma^n(Sq^2\iota_{n-6})$$

where a_1 and a_2 are mod 2 integers.

We claim that $f: \tilde{T}\tilde{E}^1 \longrightarrow \hat{E}^1$ can be altered in such a way that $(f|\tilde{T}\Omega C_1)^*(\iota_{2n-4}) = \tilde{\Psi}_{\Omega C_1}(\iota_{n-4}) = \Sigma^n(\iota_{n-4}).$

(In fact whenever such a similar situation arises this analogous deduction is always true.)

According to Peterson-Thomas [35], any other lifting of $\widetilde{Tq}_1 : \widetilde{TE}^1 \longrightarrow \widetilde{TB}$ to $\hat{E}^1$ is given by the action of a homotopy class $u \in [\widetilde{TE}^1, \Omega L_1]$ on $[\widetilde{TE}^1, \hat{E}^1]$. That is $\hat{p}_1 \circ f = \hat{p}_1 \circ f'$ iff there exists a class $u \in [\widetilde{TE}^1, \Omega L_1]$ such that

$m_*(f, u) = f'$, i.e. f' is the composite

$$\widetilde{TE}^1 \xrightarrow{d} \widetilde{TE}^1 \times \widetilde{TE}^1 \xrightarrow{u \times f} \Omega L_1 \times \hat{E}^1 \xrightarrow{m} \hat{E}^1$$

where $m : \Omega L_1 \times \hat{E}^1 \longrightarrow \hat{E}^1$ is the action of the fibre on the total space of a principal fibration and $[X, Y]$ denotes based homotopy classes of maps from X to Y. We shall use this to alter our f .

By exactness of the sequence for the pair $(\widetilde{TE}^1, \widetilde{T\Omega C}_1)$ there exist classes $z_1, z_2 \in H^{2n-4}(\widetilde{TE}^1)$ such that

$$(\widetilde{Ti}_1)^*(z_1) = \widetilde{\Psi}_{\Omega C_1}(Sq^3 \iota^*_{n-7}) \text{ and}$$
$$(Ti_1)^*(z_2) = \widetilde{\Psi}_{\Omega C_1}(Sq^2 \iota_{n-6}) .$$

Let $u \in H^{2n-4}(\widetilde{TE}^1)$ be the class $a_1 z_1 + a_2 z_2$. Consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{TE}^1 \arrow[r, "d"] & \widetilde{TE}^1 \times \widetilde{TE}^1 \arrow[r, "u \times f"] & \Omega L_1 \times \hat{E}^1 \arrow[r, "m"] & \hat{E}^1 \\
\widetilde{T\Omega C}_1 \arrow[r, "d"] \arrow[u, hook, "Ti_1"] & \widetilde{T\Omega C}_1 \times \widetilde{T\Omega C}_1 \arrow[r] \arrow[u, "Ti_1 \times Ti_1"] & \Omega L_1 \times \Omega L_1 \arrow[r, "m"] \arrow[u] & \Omega L_1 \arrow[u, "i_1"]
\end{tikzcd} \qquad (2.2.7)$$

where $m : \Omega L_1 \times \Omega L_1 \longrightarrow \Omega L_1$ is the H-space multiplication and $d : \widetilde{TE}^1 \longrightarrow \widetilde{TE}^1 \times \widetilde{TE}^1$ is the diagonal map. We claim that

$$m^*(\iota_{2n-4}) = 1 \otimes \iota_{2n-4} + \iota_{2n-4} \otimes 1 .$$

Then $(f' \mid \widetilde{T\Omega C}_1)^*(\iota_{2n-4}) = d^*(u \mid \widetilde{T\Omega C}_1 \times f)^* m^*(\iota_{2n-4})$

$$= d^*(1 \times \tilde{\imath}^*(\iota_{2n-4}) + (u \mid \widetilde{T\Omega C}_1)^*(\iota_{2n-4}) \times 1) = \widetilde{\Psi}_{\Omega C_1}(\iota_{n-4}).$$

The claim is easily derived from the commutativity of the following diagram:

$$(2.2.8)\qquad \begin{array}{ccc} \Omega K_{2n-5}\times\Omega K_{2n-5} & \xrightarrow{m} & \Omega K_{2n-5} \\ \downarrow & & \downarrow \text{ inclusion} \\ \Omega L_1\times\Omega L_1 & \xrightarrow{m} & \Omega L_1 \\ \downarrow & & \downarrow \text{ projection} \\ \Omega K_{2n-5}\times\Omega K_{2n-5} & \xrightarrow{m} & \Omega K_{2n-5} \end{array}$$

Therefore this completes the proof of Lemma 2.2.4.

2.2.9. Remark. In Lemma 2.2.4, arbitrary lifting of $\tilde{T}q_1$, $f:\tilde{T}E^1 \longrightarrow \hat{E}^1$, $f|\tilde{T}\Omega C_1$ satisfies

$$(f|\tilde{T}\Omega C_1)^*(\iota^*_{2n-7}) = \Sigma^n(\iota^*_{n-7}) \text{ and}$$

$$(f|\tilde{T}\Omega C_1)^*(\iota_{2n-6}) = \Sigma^n(\iota_{n-6}) .$$

With the footpath already set it is not hard to see that the following is true:

2.2.10. PROPOSITION. Suppose the following situation is given:

$$(2.2.11)\qquad \begin{array}{ccc} \Omega C & \xrightarrow{i} & E \\ & & \downarrow p \\ & & B & \xrightarrow{k} & C \end{array}$$

where C is a product of K_r or K^*_s , r, s > 0 , $p:E\rightarrow B$ is the principal fibration induced by the vector of cohomology classes classes $k:B\rightarrow C$. Let j be the connectivity of C. For convenience we assume that B is simply connected. Let γ be an oriented q-plane bundle over B. Apply the functor $\tilde{T}$ using the q-plane bundle to (2.2.11) to give the following

(2.2.12)
$$\begin{tikzcd}
\tilde{T}\Omega C \arrow[r, "q"] \arrow[d, "\tilde{T}i"] & \Omega L \arrow[d, "\hat{i}"] & \\
\tilde{T}E \arrow[r, "f"] \arrow[d, "\tilde{T}p"] & \hat{E} \arrow[dl, "\hat{p}"] & \\
\tilde{T}B \arrow[rr, "\tilde{\Psi}_B(k)"] & & L
\end{tikzcd}$$

where $\Omega L \longrightarrow \hat{E} \longrightarrow B$ is the principal bundle induced by $\tilde{\Psi}_B(k): \tilde{T}B \longrightarrow L$, $f: \tilde{T}E \longrightarrow \hat{E}$ is a lifting of $\tilde{T}p: \tilde{T}E \longrightarrow \tilde{T}B$, $q: \tilde{T}\Omega C \to \Omega L$ is the restriction of f to $\tilde{T}\Omega C$ and L is the corresponding product of Eilenberg-MacLane spaces formed from C by replacing the respective component K_r or K_s^* by K_{q+r} or K_{q+s}^*. Then the lifting $f: \tilde{T}E \longrightarrow \hat{E}$ of $\tilde{T}p$ can be chosen in such a way that

(2.2.13)
$$q^*(\iota_{q+r}) = \Sigma^q(\iota_r) \quad \text{and}$$
$$q^*(\iota^*_{q+s}) = \Sigma^q(\iota^*_s) \quad \text{where}$$

where $\{ \iota_{q+r}, \iota^*_{q+s} \mid r, s \text{ not all distinct} \}$ are the respective fundamental classes of the components in L.

2.2.14. We can now state the admissible class theorem which is the tool by which we identified our k-invariants.

THEOREM (Admissible Class Theorem).

Suppose the situation (2.2.1) of Proposition 2.2.10 is given. Let $\pi: A \longrightarrow B$ be a map such that $\pi^*(k)$ is homotopic to a constant map. Suppose $\theta \in H^t(E)$ $t < 2j$ is such that k is admissible for θ. Let Ω be the cohomology operation corresponding to Θ where Θ is the coset of θ with respect to $\mathrm{Ker}(q^*)\cap\mathrm{Im}(p^*) \subset H^t(E)$ and $g: A \longrightarrow E$ is a fixed lifting of π to E. Then there exists a class $e \in \Theta$ such that

$$(Tp)^*(U_B).e \in \Omega(\, (Tp)^*(U_B)) - (Tp)^*(N)$$

where U_B is the Thom class of the q-plane bundle over B and $N \subset H^{q+t}(TB)$ is such that

$$\Omega(\, (T\pi)^*(U_B)) = (T\pi)^*(N))$$

and is a coset with respect to $\mathrm{Indet}^{t+q}(TB,\Omega)$.

Proof. The proof is similar to the proof for the Generating Class Theorem. We shall give only a sketch. Let η be the collapsing functor that takes the functor T to the functor $\tilde{T}$. Consider the following commutative diagram:

(2.2.15)
$$\begin{tikzcd}
 & & & \tilde{T}\Omega C \arrow[dd, "Ti"'] \arrow[dr, "q"] & & \\
 & & & & \Omega L \arrow[dd, "i"] & \\
 & TE \arrow[rr, "\eta"] \arrow[dr, "g"] \arrow[ddr, "Tp"'] & & \tilde{T}E \arrow[dr, "f"] \arrow[ddr, "\tilde{T}p"] & & \\
 & & \hat{D} \arrow[rr, "\bar{\eta}"'] \arrow[d, "\hat{\hat{p}}"] & & \hat{E} \arrow[d, "\hat{p}"] & \\
TA \arrow[uur, "Tq"] \arrow[rr, "T\pi"] & & TB \arrow[rr, "\eta"'] & & \tilde{T}B \arrow[r, "\tilde{\Psi}_B(k)"] & L
\end{tikzcd}$$

where $\hat{p}:E \longrightarrow \tilde{T}B$ and $\hat{\hat{p}}:\hat{D} \longrightarrow TB$ are the principal fibrations induced by $\tilde{\Psi}_B(k)$ and $\Psi_B(k)$ respectively, the lifting $f:\tilde{T}E \rightarrow \hat{E}$ of $\tilde{T}p:\tilde{T}E \longrightarrow \tilde{T}B$ to $\hat{E}$ is given by Proposition 2.2.10 and $g:TE \rightarrow D$ is the 'unique' map making the diagram commute.

Let $\xi \in H^{t+q}(\hat{E})$ be a representative for the cohomology operation Ω given by Definition 2.1.1 i.e. $\mu(\xi) = \alpha.\hat{\iota}$ where $\hat{\iota} = (\iota_{i_1}, \iota_{i_2}, \ldots, \iota_{i_a})$ and ι_r is the fundamental class of the respective Eilenberg-MacLane space in ΩL. Then by 2.1.1(3) there exists a class $u \in H^{t+q}(TB)$ such that

$$(Tq)^*\left((\bar{\eta}\, g)^*(\xi) - (Tp)^*(u) \right) = 0 \quad .$$

Thus, $v = (\bar{\eta}\, g)^*(\xi) - (Tp)^*(u) \in \mathrm{Ker}((Tq)^*)$. Using the Thom isomorphism Ψ_E there exists a unique class $e \in H^t(E)$ such that $\Psi_E(e) = v$. It is not difficult by looking into the construction of μ to deduce that

(2.2.16) $\qquad \mu(e) = \mu(\theta)$.

Hence we observe that e is in the coset of θ with respect to $\mathrm{Im}(p^*) \cap \mathrm{Ker}(q^*)$ since $\mathrm{Ker}(\mu) = \mathrm{Im}(p^*)$ in dimensions $< 2j$.

Our main application of Theorem 2.2.14 is the following:

2.2.17. THEOREM. Let the 2-stage Postnikov tower (2.1.4) be given. Let the k-invariants for the first and second stages be k^1 and k^2 given by 1.2.10 and Table 1.2.13. Let Φ_i^* , i= 1, 2,..,6 be the secondary cohomology operations (on integral classes) defined by Table 2.1.5. Then

$$(Tq_1)^*(U_{BSpin_n}).k_i^2 \in \Phi_i^*((Tq_1)^*(U_{BSpin_n})) - (Tq_1)^*(N_i)$$

$$i = 1, 2, 3, 4, 5 \text{ and } 6$$

where U_{BSpin_n} is the Thom class of the universal spin n-plane bundle and $\{ N_i \}_{1 \leq i \leq 6}$ are the respective coset with respect to $\mathrm{Indet}^*(TBSpin_n, \Phi_i^*)$ such that

$$\Phi_i^*((T\pi)^*(U_{BSpin_n})) = (T\pi)^*(N_i) \quad i = 1, 2, \ldots, 6 \ .$$

Proof. By Proposition 2.1.6, Theorem 2.2.14 and the observation that $\mathrm{Ker}(p_1^*) \cap \mathrm{Im}(q_1^*)$ is zero in dimensions $<$ $2(n-7)$.

2.2.18. COROLLARY. Hypothesis and notation as in Theorem 2.2.14. Suppose $\zeta : X \longrightarrow B$ is a map from a space X to B such that $\zeta^*(k) \simeq *$ the constant map. Then there exists a class $e \in Ⓜ$ and a class $u \in H^t(B)$ such that

$$(T\pi)^*(U_B) . (e(\zeta) + \zeta^* u) = \Omega((T\pi)^*(U_B)) .$$

Proof. Immediate from Theorem 2.2.14.

2.2.19. COROLLARY. Suppose X is a space and $\zeta : X \longrightarrow BSpin_n$ is the classifying map of a spin n-plane bundle over X satisfying $\zeta^*(k^1_i) = 0$ for i = 1, 2, 3 where $k^1 = (k^1_1, k^1_2, k^1_3)$ is the k-invariant for the first stage of an n-MPT for (2.1.2) . Then there exist classes $\{ u_i \}_{1 \leqslant i \leqslant 6}$ such that

$$U_\zeta . (k^2_i(\zeta) + \zeta^*(u_i)) = \Phi^*_i(U_\zeta) , \quad 1 \leqslant i \leqslant 6$$

where U_ζ is the Thom class of the bundle ζ, $\{ \Phi^*_i \}_{1 \leqslant i \leqslant 6}$ are the secondary cohomology operations given by Table 2.1.5, $k^2 = (k^2_1, k^2_2, k^2_3, \ldots, k^2_6)$ is the k-invariant for the second stage of the n-MPT for (2.1.2); $\{ u_i \}_{1 \leqslant i \leqslant 6}$ is independent of ζ and is determined by $\{ \Phi^*_i((T\pi)^*(U_{BSpin_n})) \}_{1 \leqslant i \leqslant 6}$.

Proof. Immediate from Theorem 2.2.17.

CHAPTER 3. SECONDARY COHOMOLOGY OPERATIONS AND THE n-MPT FOR $V_{n,7} \longrightarrow BSPin_{n-7} \longrightarrow BSPin_n$

This chapter relies heavily on Mahowald-Peterson [25] and Hughes-Thomas [34] . Nothing virtually new is proved except that the computation of Φ_i^* is presented in the flavour of the view point of lifting. It provides the ease and the confidence to identify the k-invariant for the third stage of the n-MPT for the fibration $V_{n,7} \longrightarrow BSPin_{n-7} \longrightarrow BSPin_n$.

§ 3.1. Preliminaries

We collect in this section all the well-known facts which we need without proofs. Let X in this section be a pointed space whose base point possesses a contractible neighbourhood.

FACTS:

3.1.1. If X is q-connected then the join $\Omega X * \Omega X$ is 2q-connected where ΩX denotes the based loops on X.

3.1.2. THEOREM (Barcus-Meyer [7]). Let X be q-connected $q > 2$ and let $h: \Sigma\Omega X \longrightarrow X$ be the adjunct to $1_{\Omega X}$. h regarded as a fibration in the standard way can be written as follows:

(3.1.3) $\qquad \Omega X * \Omega X \longrightarrow \Sigma\Omega X \longrightarrow X$

where by abuse of notation we write the homotopy types of the spaces involved. Then

(1) the following diagram is commutative for all $i > 1$:

$$(3.1.4)\qquad \begin{tikzcd} H^i(\Omega X * \Omega X) & H^i(\Sigma\Omega X) \arrow[l, "i^*"'] \arrow[d, "s"] & H^i(X) \arrow[l, "h^*"'] \arrow[dl, "\sigma"] \\ H^{i-1}(\Omega X \times \Omega X) \arrow[u, "\Delta"] & H^{i-1}(\Omega X) \arrow[l, "m"'] & \end{tikzcd}$$

where H is any singular cohomology theory with coefficients in a principal ideal ring, $\sigma: H^i(X) \longrightarrow H^{i-1}(\Omega X)$ is the loop homomorphism, $m: \Omega X \times \Omega X \longrightarrow \Omega X$ is the Hopf multiplication, $s: H^*(\Sigma\Omega X) \longrightarrow H^{*-1}(\Omega X)$ is the suspension isomorphism and $\Delta: H^*(\Omega X \times \Omega X) \longrightarrow H^{*+1}(\Omega X * \Omega X)$ is the Mayer-Vietoris coboundary for the triad $[\Omega X * \Omega X, A_1, A_2]$ where $A_1 \cap A_2$ is homeomorphic to $\Omega X \times \Omega X$ and the deformation retracts of A_1 and A_2 are $\Omega X \times 0$ and $1 \times \Omega X$ respectively; and

(2) for $1 < i \leq 4q+2$ there exists homomorphisms $\theta^*: H^i(X \times X, X \vee X) \longrightarrow H^{i-1}(\Omega X * \Omega X)$ such that the following diagram commutes

$$(3.1.5)\qquad \begin{tikzcd} H^i(X) & H^{i-1}(\Omega X * \Omega X) \arrow[l, "\tau"'] \\ H^i(X) \arrow[u, equal] & H^i(X \times X, X \vee X) \arrow[l, "d^*"'] \arrow[u, "\theta^*"'] \end{tikzcd}$$

where $\tau: H^{i-1}(\Omega X * \Omega X) \longrightarrow H^i(X)$ is the transgression in the fibration (3.1.3) if $q \geq 2$, $d: H^i(X \times X, X \vee X) \longrightarrow H^i(X)$ is the diagonal approximation. Moreover θ^* is an isomorphism for $i < 3q+1$ and a monomorphism for $i = 3q+1$.

3.1.6. PROPOSITION. Let π be a finitely generated Abelian group. Then

(1) $\pi_j(\Sigma(K(\pi,q))) \cong 0$ for $j < q+1$; $\pi_{q+1}(\Sigma(K(\pi,q))) \cong \pi$;

$$\pi_{2q+1}(\Sigma(K(\pi,q))) \cong \pi \otimes \pi;$$

$$\pi_{2q+2}(\Sigma(K(\pi,q))) \cong \pi * \pi; \text{ and}$$

(2) the Moore-Postnikov invariant

$$k^{i+1}(K(\pi,q)*K(\pi,q)) = 0 \quad \text{for } i \leq 3q$$

Hence all the k-invariants of dimensions $\leq 3q+1$ for the fibration $K(\pi,q)*K(\pi,q) \to \Sigma(K(\pi,q)) \longrightarrow K(\pi,q+1)$ come from the cohomology of the base $K(\pi,q+1)$. In particular,

$$k^{2q+2}(\Sigma(K(\pi,q))) = -\iota \cup \iota \in H^{2q+2}(K(\pi,q+1), \pi \otimes \pi)$$

where $\iota \in H^{q+1}(K(\pi,q+1);\pi)$ is the fundamental class of $K(\pi,q+1)$.

The following lemma is well known:

3.1.7. LEMMA. Suppose we have the following splitting of a vector bundle γ over a space B: $\gamma \sim \xi \oplus \epsilon$ where ϵ is the trivial line bundle. Then there is a natural map

$$e: \Sigma(T(\xi)) \longrightarrow T(\gamma)$$

which is a homotopy equivalence. Let $s: H^*(\Sigma(T(\xi))) \longrightarrow H^{*-1}(T(\xi))$ be the suspension isomorphism where H is any singular cohomology theory with coefficients in a principal ideal ring with a 'cup' product pairing . Then

$$s(e^*(U_\gamma . x)) = U_\xi . x \quad \text{for all } x \in H^*(B).$$

The proof of Lemma 3.1.7 follows from some complicated argument, essentially due to Milnor, using the stability property of cup product. We leave the details to the reader.

We shall need the next three technical lemmas for the computation of operations of type Φ_3^*, Φ_5^* or Φ_6^*. First we make some definitions.

3.1.8. Definitions. Consider the following situation:

(3.1.9)
$$\begin{tikzcd} \Omega C \arrow[r, "i"] & E \arrow[d, "p"] & \\ & B \arrow[r, "w"] & C \end{tikzcd}$$

where $p:E \to B$ is the principal fibration induced by $w:B \to C$. Then we have the following fibre square:

(3.1.10)
$$\begin{tikzcd} \Omega C\times E \arrow[r, "\nu"] \arrow[d, "proj."'] & E \arrow[d, "p"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

Replacing B by the mapping cylinder of p we consider $p:E \to B$ as an inclusion and so we have the map

$$p_1: \quad (E, \Omega C\times E) \longrightarrow (B, E)$$

where E is to be thought of as the mapping cylinder of ν and B is to be thought of as the mapping cylinder of p .

We make the following definitions:

$$\Lambda^* = d^*(\mathrm{Ker}(p_1^*)) ;$$
$$T^*(\Omega C\times E) = \delta^{-1}\, p_1^*(H^*(B,E));$$
$$T^*(\Omega C\times E, E) = j^{*\,-1} T^*(\Omega C\times E) ; \quad \text{and}$$
$$T^*(\Omega C\wedge E) = c^{*-1} T^*(\Omega C\times E)$$

where $\delta: H^*(\Omega C\times E) \longrightarrow H^{*+1}(E, \Omega C\times E)$ is the coboundary operator for the pair $(E, \Omega C\times E)$ and $c:\Omega C\times E \longrightarrow \Omega C\wedge E$ is the collapsing map and $j:\Omega C\times E \longrightarrow (\Omega C\times E, E)$, $d:B \longrightarrow (B,E)$ are the respective inclusions.

3.1.11. LEMMA (E. Thomas). Data as given by (3.1.9). Suppose B is m-connected and C is q-connected. Then if $u, v \in H^*(E)$ are cohomology classes of dimensions $\leq m+q$ and $i^*(v) = 0$, $\mu(u.v) = i^*(u)\otimes v \in H^*(\Omega C \wedge E)$ where $\mu : H^*(E) \longrightarrow H^*(\Omega C \wedge E)$ is defined analogously as in §1.3.2.

3.1.12. LEMMA (E. Thomas). Suppose $C = K(Z_2, q+1)$. Then $\quad T^i(\Omega C \wedge E) = H^i(\Omega C \wedge E)$ for $0 \leq i \leq 2q+m+1$.

3.1.13. LEMMA (E. Thomas).

(1) The sequence

$$(3.1.14) \qquad H^*(E) \cap \mathrm{Ker}(i^*) \xrightarrow{\ \mu\ } T^*(\Omega C \wedge E) \xrightarrow{\ \tau\ } H^{*+1}(B)/\Lambda^*$$

is exact; suppose $C = K(Z_2, q+1)$, then

(2) $\Lambda^{2q+2} = \{ v^2 \} \subset H^{2q+2}(B)$ and

(3) the sequence

$$(3.1.15) \qquad H^{2q+1}(B) \xrightarrow{\ p^*\ } H^{2q+1}(E) \cap \mathrm{Ker}(i^*) \xrightarrow{\ \mu\ } T^{2q+1}(\Omega C \wedge E)$$

is exact.

The proofs of these lemmas can be found in Thomas [36] and we refer the reader to it for details.

The rest of this chapter is organised as follows: In sections 2, 3 and 4 the computation of operations of types Φ_3^*, Φ_5^* and Φ_6^* is carried out with emphasis on their stability property and in section 5 we apply the results to our operations and thus gives us the indication to obtain some sort of an identification of the n-dimensional component of the k-invariant for the third stage of the n-MPT for (2.1.2) .

§ 3.2. Operations of Type Φ_1^*

Adem gave the following relations:

$$Sq^2 Sq^{q+1} + Sq^{q+2} Sq^1 = 0 \quad \text{for } q \equiv 0,1 \bmod 4 \tag{3.2.1}$$

and for $q \equiv 0 \bmod 4$ we can write the relation (3.2.1) as

$$Sq^2(\delta Sq^q) + Sq^{q+2} Sq^1 = 0 . \tag{3.2.2}$$

We shall prove the following theorem:

3.2.3. THEOREM (Mahowald-Peterson)

There exists a stable secondary cohomology operation Φ_1' associated with the relation (3.2.1) such that

$$b_q \cup Sq^2 b_q \in \Phi_1'(b_q)$$

where b_q is the fundamental class of Y_q the universal example for q-dimensional mod 2 cohomology class x satisfying $Sq^1 x = 0$.

3.2.4. COROLLARY. There exists a stable secondary cohomology operation Φ_1 associated with the relation (3.2.2) such that

$$Sq^2 b_q \cup b_q \in \Phi_1(b_q)$$

where b_q is the fundamental class of Y_q .

3.2.5. COROLLARY. There exists a stable secondary cohomology operation $\Phi_1'^*$ associated with the relation

$$Sq^2 Sq^{q+1} = 0 \text{ on integral classes for } q \equiv 0,1(4) \tag{3.2.6}$$

such that

$$\iota_q^* \cup Sq^2 \iota_q^* \in \Phi_1'^*(\iota_q^*) .$$

Corollary 3.2.5 follows from Theorem 3.2.3 and Corollary 3.2.4 follows almost in the same manner as 3.2.5 from Theorem 3.2.3. The details are left to the reader.

3.2.5. Definition. The following commutative diagram of spaces and maps shall often be refered to as the first stage Moore-Postnikov decomposition of $h: \Sigma K_q^* \longrightarrow K_{q+1}^*$:

(3.2.6)
$$\begin{tikzcd} & K_{2q+1}^* \arrow[rr, "i_{q+1}"] & & G_{q+1} \arrow[d, "p_{q+1}"] & \\ K_q^* * K_q^* \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma K_q^* \arrow[ur, "s_1"] \arrow[r, "h"] & K_{q+1}^* \arrow[r, "\iota_{q+1}^{*2}"] & K_{2q+2}^* \end{tikzcd}$$

where $p_{q+1}: G_{q+1} \longrightarrow K_{q+1}^*$ is the principal fibration over K_{q+1}^* induced by $\iota_{q+1}^* \cup \iota_{q+1}^*$, $s_1: \Sigma K_q^* \longrightarrow G_{q+1}$ is a lifting of $h: \Sigma K_q^* \longrightarrow K_{q+1}^*$ to G_{q+1} and s_2 is the restriction of s_1 to $K_q^* * K_q^*$.

3.2.7. We shall rephrase Theorem 3.2.3 in the form that is easier to understand. In fact we shall proved Corollary 3.2.4 and then deduce Theorem 3.2.3 from it.

PROPOSITION. Let G_{q+1} be the universal example for integral cohomology class x of dimension q+1 satisfying $x^2 = 0$. That is G_{q+1} is given by (3.2.6) by 3.1.6(2). Given a cohomology class $v \in H^{2q+3}(G_{q+1})$ such that $i_{q+1}^*(v) = Sq^2 \iota_{2q+1}^*$.

Then $\qquad s_1^*(v) = \gamma + h^*(y)$

where $\gamma \in H^{2q+3}(\Sigma K_q^*)$ is such that $s(\gamma) = Sq^2 \iota_q^* \cup \iota_q^*$, and $y = \sigma^1(x)$ for some $x \in H^{2q+4}(K_{q+2}^*)$.

In particular if $q \equiv 0$ (4) then γ uniquely determines a stable secondary cohomology operation (on integral classes) Φ_1^* associated with the relation

(3.2.8) $\qquad Sq^2(\delta Sq^q) = 0$ on integral classes for $q \equiv 0(4)$ modulo the stable primary cohomology operation Sq^{q+2}.

Proof. Let $\gamma \in H^{2q+3}(\Sigma K_q^*)$ be such that $s(\gamma) = Sq^2\iota_q^* \cup \iota_q^*$. Then by Theorem 3.1.2(1) and letting $a*b = \Delta(a \times b)$ for $a, b \in H^*(K_q^*)$,

$$
\begin{aligned}
i^*(\gamma) &= \Delta\, m^*(Sq^2\iota_q^* \cup \iota_q^*) \\
&= \Delta\,(Sq^2\iota_q^* \cdot \iota_q^* \times 1 + 1 \times Sq^2\iota_q^* \cdot \iota_q^* + Sq^2\iota_q^* \times \iota_q^* \\
&\qquad + \iota_q^* \times Sq^2\iota_q^*) \\
&= \iota_q^* * Sq^2\iota_q^* + \iota_q^* * Sq^2\iota_q^* .
\end{aligned}
$$

By Theorem 3.1.2(2) and the Serre exact sequence for the fibration $p_{q+1} : G_{q+1} \longrightarrow K_{q+1}^*$ we see that

$$s_2^*(\iota_{2q+1}^*) = \iota_q^* * \iota_q^* \quad .$$

Hence $i^* \circ s_1^*(v) = s_2^*(Sq^2\iota_{2q+1}^*) = Sq^2\iota_q^* * \iota_q^* + \iota_q^* * Sq^2\iota_q^*$,

so that $\qquad i^*(s_1^*(v) - \gamma) = 0$.

Therefore by exactness of the Serre exact sequence for h

$$s_1^*(v) = \gamma + h^*(y)$$

where $y \in H^{2q+3}(K_{q+1}^*)$ can be written as $\mathbb{Z}$-decomposable. A little reflection on the mod 2 cohomology groups of K_{q+1}^* tells that there exists $x \in H^{2q+4}(K_{q+2}^*)$ modulo $Sq^{q+2}\iota_{q+2}^*$ such that

$$\sigma^1(x) = y \ .$$

This proves the first part of Proposition 3.2.7. To see the second assertion consider the following commutative diagram for $q \equiv 0\ (4)$:

$$
\begin{array}{ccccccccc}
0 \Rightarrow & H^{2q+4+s}(K_{q+2+s}^*) & \rightarrow & H^{2q+4+s}(G_{q+2+s}) & \longrightarrow & H^{2q+4+s}(K_{2q+2+s}^*) & \rightarrow 0 \\
& \downarrow{\sigma^s} & & \downarrow{\sigma^s} & & \downarrow{\sigma^s} & \\
0 \rightarrow & H^{2q+4}(K_{q+2}^*) & \longrightarrow & H^{2q+4}(G_{q+2}) & \longrightarrow & H^{2q+4}(K_{2q+2}^*) & \longrightarrow 0
\end{array}
\tag{3.2.9}
$$

where by abuse of notation we denote the universal example for $(q+2+s)$-dimensional integral cohomology class x satisfying $Sq^q x = 0$ by G_{q+2+s}.

The rows are easily seen to be exact. The first and last 'vertical' homomorphisms are obviously isomorphisms. Therefore the 5-Lemma applies to give that $\sigma^s : H^{2q+4+s}(G_{q+2+s}) \longrightarrow H^{2q+4}(G_{q+2})$ is an isomorphism. Thus take a representative $v \in H^{2q+4}(G_{q+2})$ then our previous argument gives that

$$s_1^*\sigma(v) = \gamma + h^*(y)$$

where $y = \sigma^1(x)$. Define $\phi = v - p_{q+2}^*(x)$; then

$$s_1^*\sigma(\phi) = \gamma \ .$$

This completes the proof of Proposition 3.2.7.

We have an immediate corollary:

3.2.10. COROLLARY. For any lifting $s_1 : \Sigma K_q^* \longrightarrow G_{q+1}$ of $h : \Sigma K_q^* \longrightarrow K_{q+1}^*$ to G_{q+1} $s_1^* : H^{2q+3}(G_{q+1}) \rightarrow H^{2q+3}(\Sigma K_q^*)$ is a monomorphism.

3.2.11. Proof of Theorem 3.2.3.

Let $\tilde{G}_s$ be the universal example for mod 2 s-dimensional cohomology class x satisfying $Sq^1 x = Sq^{q+1} x = 0$. Then we have the following commutative diagram:

(3.2.12)
$$\begin{tikzcd}
 & & & \tilde{G}_{q+1} \arrow[dd, "\tilde{p}_{q+1}"] \\
 & G_{q+1} \arrow[urr, "\bar{\rho}"] \arrow[dd, "p_{q+1}"] & & \\
 & & \Sigma Y_q \arrow[uur, "s_1"] \arrow[r, "\tilde{h}"] & Y_{q+1} \arrow[d] \\
\Sigma K_q^* \arrow[uur, "s_1"] \arrow[r, "h"] \arrow[urr, "\Sigma\rho"] & K_{q+1}^* \arrow[urr, "\rho"] \arrow[rr, "\rho"] & & K_{q+1}
\end{tikzcd}$$

where $\rho : K^*_{q+1} \longrightarrow K_{q+1}$ is mod 2 reduction and $\tilde{G}_{q+1}$ is regarded as the total space of the principal fibration over Y_{q+1} induced by $Sq^{q+1} b_{q+1}$ (where b_{q+1} is the fundamental class of Y_{q+1}) and the rest of the diagram is composed of the obvious liftings that make it commutative.

Therefore given $v \in H^{2q+3}(G_{q+1})$ such that $i^*(v) = Sq^2 \iota_{2q+1}$

$$\begin{aligned}(\Sigma\rho)^* s_1^*(v) &= s_1^* \bar{\rho}^*(v)\\ &= \gamma + h^*\sigma(\tilde{x}) \quad \text{by Proposition 3.2.7}\\ &\qquad \text{for some } \tilde{x} \in H^{2q+4}(K^*_{q+2})\\ &= (\Sigma\rho)^*(\tilde{\gamma}) + h^*\rho^*\sigma(z) \quad \text{for } \rho^* \text{ is epimorphic}\\ &= (\Sigma\rho)^*(\tilde{\gamma} + \tilde{h}^*\sigma(z)) \text{ by commutativity of (3.2.12)}\end{aligned}$$

Thus, $\quad s_1^*(v) = \tilde{\gamma} + \tilde{h}^*\sigma(z) + \approx{x}$

where $s(\tilde{\gamma}) = Sq^2 b_q \cup b_q$ and $\hat{\approx{x}} \in \operatorname{Ker}((\Sigma\rho)^*) \cap H^{2q+3}(\Sigma Y_q)$.

Since Y_q is of the same homotopy type as $K(Z_4, q)$ we see that $\operatorname{Ker}((\Sigma\rho)^*)$ is the ideal generated by

$\{ Sq^I \psi \mid I = (i_1, \ldots, i_r)$ admissible with $i_r > 1$ and ψ is the unique representative for the stable operation associated with $Sq^1 Sq^1 = 0 \}$.

Hence there exists $y \in H^{2q+4}(Y_{q+2})$ such that

$$\approx{x} = \tilde{h}^*\sigma(y) .$$

Thus we have shown that there exists $x \in H^{2q+4}(Y_{q+2})$ such that

(3.2.13) $\quad s_1^*(v) = \gamma + \tilde{h}^*(\sigma(x) + d)$ where d is decomposable.

We have a diagram analogous to (3.2.9) with exact rows and the cohomology groups on the left are replaced by those of the Y_j's.

Using a similar argument as in the proof of the analogous statement in the proof of Proposition 3.2.7, it is shown that

$$\sigma^s : H^{2q+4+s}(G_{q+2+s}) \longrightarrow H^{2q+4}(G_{q+2})$$

is an isomorphism. Now let $\phi \in H^{2q+4}(G_{q+2})$ be a representative for the operation associated with (3.2.1) then by (3.2.13)

$$s_1^* \sigma(\phi) = \tilde{\gamma} + \tilde{h}^*(\sigma(x) + \text{decomposable})$$

$$\text{for some } x \in H^{2q+4}(Y_{q+2}).$$

Let $\tilde{\phi} = \phi - p_{q+2}{}^*(x)$. Thus $s \cdot s_1^* \sigma(\tilde{\phi}) = Sq^2 b_q \cup b_q$. And so $\tilde{\phi} \in H^{2q+4}(G_{q+2})$ is a natural candidate representing a stable secondary cohomology operation associated with the relation (3.2.1) satisfying the conclusion of Theorem 3.2.3. This completes the proof.

§3.3. Operations of Type Φ_4^*

For $q \equiv 2 \bmod 4$ Adem gave the following relations in $\mathfrak{A}(2)$:

(3.3.1) $\quad (Sq^2 Sq^1) Sq^q + Sq^1 Sq^{q+2} + Sq^{q+2} Sq^1 = 0 \quad ;$

(3.3.2) $\quad Sq^2 (\delta Sq^q) + Sq^1 Sq^{q+2} + Sq^{q+2} Sq^1 = 0 \quad ;$

and on integral classes the following:

(3.3.1)* $\quad (Sq^2 Sq^1) Sq^q + Sq^1 Sq^{q+2} = 0 \quad$ and

(3.3.2)* $\quad Sq^2 (\delta Sq^q) + Sq^1 Sq^{q+2} = 0$.

We use the same notation as in §3.2. Similar to §3.2 we obtain the following theorem:

3.3.3. THEOREM (Mahowald-Peterson).

(1) There exists a stable secondary cohomology operation Ψ_4 associated with the relation (3.3.2) such that

$$b_q \cup Sq^2 b_q \in \Psi_4(b_q)$$

where b_q is the fundamental class of Y_q the universal example for mod 2 cohomology class x of dimension q satisfying $Sq^1 x = 0$;

(2) there exists a stable secondary cohomology operation Ψ_4^* associated with the relation $(3.3.2)^*$ such that

$$\iota_q^* \cup Sq^2 \iota_q^* \in \Psi_4^*(\iota_q^*)$$

where ι_q^* is the fundamental class of $K(Z,q)$;

(3) there exist stable secondary cohomology operations Φ_4 and Φ_4^* associated with the relations (3.3.1) and $(3.3.1)^*$ respectively such that $\Phi_4 \subset \Psi_4$, $\Phi_4^* \subset \Psi_4^*$ where Ψ_4 and Ψ_4^* are given by (1) and (2); hence

$$0 \in \Phi_4(b_{q-1}) \text{ , } 0 \in \Phi_4^*(\iota_{q-1}^*) \text{ .}$$

Proof. The proof of parts (1) and (2) is similar to the proof of Theorem 3.2.3; part (3) follows from parts (1) and (2).

§ 3.4. Operations of Type Φ_3^* , Φ_5^* or Φ_6^*.

Consider the following Adem's relation:

(3.4.1) $\quad Sq^4 Sq^{q+1} + Sq^{q+4} Sq^1 + Sq^{q+3} Sq^2 = 0$ for $q \equiv 0,3$ (8);

(3.4.2) $\quad Sq^4(\delta Sq^q) + Sq^{q+3} Sq^2 + Sq^{q+4} Sq^1 = 0$ for $q \equiv 0$ (8);

(3.4.3) $\quad Sq^4 Sq^{q+1} + Sq^{q+3} Sq^2 = 0$ for $q \equiv 1,2$ (8) and

(3.4.4) $\quad Sq^4(\delta Sq^q) + Sq^{q+3} Sq^2 = 0$ for $q \equiv 2$ (8).

We also have the corresponding relations on integral classes $(3.4.1)^*$, $(3.4.2)^*$, $(3.4.3)^*$ and $(3.4.4)^*$ respectively.

We may now state the main theorem of this section:

3.4.5. THEOREM(Mahowald-Peterson-Hughes-Thomas).

(1) There exist stable secondary cohomology operations Φ_6^*, Φ_3^*, Φ_5^* and Ψ_5^* associated with the relations $(3.4.1)^*$, $(3.4.2)^*$, $(3.4.3)^*$ and $(3.4.4)^*$ respectively such that

$$d_q^* \cup Sq^4 d_q^* \in \Phi_i^*(d_q^*) \quad i = 6,3, \text{ or } 5 \text{ and}$$

$$d_q^* \cup Sq^4 d_q^* \in \Psi_5^*(d_q^*)$$

where d_q^* is the fundamental class of D_q^* the universal example for q-dimensional integral cohomology class x satisfying $Sq^2x = 0$;

(2) there exist stable secondary cohomology operations Φ_6, $\overline{\Phi}_3$, Φ_5 and Ψ_5 associated with the relations (3.4.1), (3.4.2), (3.4.3) and (3.4.4) such that

$$d_q \cup Sq^4 d_q \in \Phi_i(d_q) \quad i = 6,\ 3\ ;$$

$$e_q \cup Sq^4 e_q \in \Phi_5(e_q) \text{ and } e_q \cup Sq^4 e_q \in \Psi_5(e_q)$$

where d_q is the fundamental class of D_q the universal example for q-dimensional mod 2 cohomology class x satisfying $Sq^1x = Sq^2x = 0$ and e_q is the fundamental class of $\tilde{E}_q$ the universal example for q-dimensional mod 2 cohomology class x satisfying $Sq^2x = 0$.

We shall need the following propositions in the proof of Theorem 3.4.5. First we set up some notations:

3.4.6. The following commutative diagrams shall be referred to in the next three propositions:

(3.4.7)
$$\begin{tikzcd} & K^*_{2q+1} \arrow[rr, "i_{q+1}"] & & G^*_{q+1} \arrow[d, "p_{q+1}"] & \\ D^*_q * D^*_q \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma D^*_q \arrow[ur, "s_1"] \arrow[r, "h"] & D^*_{q+1} \arrow[r, "(d_{q+1})^2"] & K^*_{2q+2} \end{tikzcd}$$

where D^*_{q+1} is the universal example for $(q+1)$-dimensional integral cohomology class x satisfying $Sq^2x = 0$, $p_{q+1}: G^*_{q+1} \longrightarrow D^*_{q+1}$ is the principal fibration over D^*_{q+1} induced by $d^*_{q+1} \cup d^*_{q+1}$ where d^*_{q+1} is the fundamental class of D^*_{q+1} and the rest of the diagram is made up of the obvious maps and spaces;

(3.4.8)
$$\begin{tikzcd} & K_{2q+1} \arrow[rr, "i_{q+1}"] & & G_{q+1} \arrow[d, "p_{q+1}"] & \\ D_q * D_q \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma D_q \arrow[ur, "s_1"] \arrow[r, "h"] & D_{q+1} \arrow[r, "Sq^{q+1}d_{q+1}"] & K_{2q+2} \end{tikzcd}$$

where D_{q+1} is the universal example for $(q+1)$-dimensional mod 2 cohomology class x satisfying $Sq^1x = Sq^2x = 0$, $p_{q+1}: G_{q+1} \longrightarrow D_{q+1}$ is the principal fibration over D_{q+1} induced by $Sq^{q+1}d_{q+1}$ where d_{q+1} is the fundamental class of D_{q+1}; and for $q \equiv 1,2 \ (8)$

(3.4.9)
$$\begin{tikzcd} & K_{2q+1} \arrow[rr, "\tilde{i}_{q+1}"] & & \tilde{G}_{q+1} \arrow[d, "\tilde{p}_{q+1}"] & \\ \tilde{D}_q * \tilde{D}_q \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma \tilde{D}_q \arrow[ur, "s_1"] \arrow[r, "h"] & \tilde{D}_{q+1} \arrow[r, "Sq^{q+1}e_{q+1}"] & K_{2q+2} \end{tikzcd}$$

where $\tilde{D}_{q+1}$ is the universal example for $(q+1)$-dimensional mod 2 cohomology class x satisfying $Sq^2x = 0$, $p_{q+1}: \tilde{G}_{q+1} \longrightarrow \tilde{D}_{q+1}$ is the principal fibration over $\tilde{D}_{q+1}$ induced by $Sq^{q+1}e_{q+1}$ where e_{q+1} is the fundamental class of $\tilde{D}_{q+1}$.

We shall need the following propositions in the proof of Theorem 3.4.5. First we set up some notations:

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D^*_q * D^*_q \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma D^*_q \arrow[ur, "s_1"] \arrow[r, "h"] & D^*_{q+1} \arrow[r, "(d_{q+1})^2"] & K^*_{2q+2}
\end{tikzcd}$$

where D^*_{q+1} is the universal example for $(q+1)$-dimensional integral cohomology class x satisfying $Sq^2 x = 0$, $p_{q+1}: G^*_{q+1} \longrightarrow D^*_{q+1}$ is the principal fibration over D^*_{q+1} induced by $d^*_{q+1} \cup d^*_{q+1}$ where d^*_{q+1} is the fundamental class of D^*_{q+1} and the rest of the diagram is made up of the obvious maps and spaces;

(3.4.8)
$$\begin{tikzcd}
 & K_{2q+1} \arrow[rr, "i_{q+1}"] & & G_{q+1} \arrow[d, "p_{q+1}"] & \\
D_q * D_q \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma D_q \arrow[ur, "s_1"] \arrow[r, "h"] & D_{q+1} \arrow[r, "Sq^{q+1} d_{q+1}"] & K_{2q+2}
\end{tikzcd}$$

where D_{q+1} is the universal example for $(q+1)$-dimensional mod 2 cohomology class x satisfying $Sq^1 x = Sq^2 x = 0$, $p_{q+1}: G_{q+1} \longrightarrow D_{q+1}$ is the principal fibration over D_{q+1} induced by $Sq^{q+1} d_{q+1}$ where d_{q+1} is the fundamental class of D_{q+1}; and for $q \equiv 1,2\ (8)$

(3.4.9)
$$\begin{tikzcd}
 & K_{2q+1} \arrow[rr, "\tilde{i}_{q+1}"] & & \tilde{G}_{q+1} \arrow[d, "\tilde{p}_{q+1}"] & \\
\tilde{D}_q * \tilde{D}_q \arrow[ur, "s_2"] \arrow[rr, "i"] & & \Sigma \tilde{D}_q \arrow[ur, "s_1"] \arrow[r, "h"] & \tilde{D}_{q+1} \arrow[r, "Sq^{q+1} e_{q+1}"] & K_{2q+2}
\end{tikzcd}$$

where $\tilde{D}_{q+1}$ is the universal example for $(q+1)$-dimensional mod 2 cohomology class x satisfying $Sq^2 x = 0$, $p_{q+1}: \tilde{G}_{q+1} \longrightarrow \tilde{D}_{q+1}$ is the principal fibration over $\tilde{D}_{q+1}$ induced by $Sq^{q+1} e_{q+1}$ where e_{q+1} is the fundamental class of $\tilde{D}_{q+1}$.

We can now state our propositions:

3.4.10. PROPOSITION. Suppose $\phi \in H^{2q+5}(G^*_{q+1})$ is such that $i^*_{q+1}(\phi) = Sq^4\iota^*_{2q+1}$. Then

$$s_1^*(\phi) = \gamma^* + h^*(x)$$

where $\gamma^* \in H^{2q+5}(\Sigma D^*_q)$ is such that $s(\gamma^*) = d^*_q \cup Sq^4 d^*_q$ and $x \in H^{2q+5}(D^*_{q+1})$ is given by

$$x = d + p^*\sigma^3(y) \quad \text{for some } y \in H^{2q+8}(K^*_{q+4})$$

where $d \in H^{2q+5}(D^*_{q+1})$ is decomposable and $p: D^*_{q+1} \longrightarrow K^*_{q+1}$ is the principal fibration defining D^*_{q+1} .

3.4.11. PROPOSITION. Let $\phi \in H^{2q+5}(G_{q+1})$ be such that $i^*_{q+1}(\phi) = Sq^4\iota_{2q+1}$. Then

$$s_1^*(\phi) = \gamma + h^*(x)$$

where $\gamma \in H^{2q+5}(\Sigma D_q)$ is the unique class such that

$$s(\gamma) = d_q \cup Sq^4 d_q \quad \text{and}$$

$x \in H^{2q+5}(D_{q+1})$ is given by

$$x = d + p^*\sigma^3(y) \text{ for some } y \in H^{2q+8}(Y_{q+4})$$

where $d \in H^{2q+5}(D_{q+1})$ is decomposable and $p: D_{q+1} \longrightarrow Y_{q+1}$ is the principal fibration over Y_{q+1} defining D_{q+1} .

Before we give the proofs for these two propositions we shall prove Theorem 3.4.5 assuming that Propositions 3.4.10 and 3.4.11 are true.

3.4.12. Proof of Theorem 3.4.5.(1)

Let $v \in H^{2q+8}(G^*_{q+4})$ be a representative for a stable secondary cohomology operation associated with the relation $(3.4.2)^*$ or $(3.4.4)^*$ such that $i^*_{q+4}(v) = Sq^4 \iota^*_{2q+4}$. Then by Proposition 3.4.10

$$s_1^* \sigma^3(v) = \gamma^* + h^* \, p^* \sigma^3(y) \text{ for some } y \in H^{2q+8}(K^*_{q+4}).$$

Define $\tilde{\phi} = v - p^*_{q+4} \, p^*(y)$. Then

$$s \, s_1^* \sigma^3(\tilde{\phi}) = s(\gamma^* + h^* \, p^* \sigma^3(y) - s_1^* \, p^*_{q+1} p^* \sigma^3(y) \,)$$

$$= s(\gamma^*) = d^*_q \cup Sq^4 d^*_q \,.$$

Thus $\tilde{\phi} \in H^{2q+8}(G^*_{q+4})$ is a natural candidate for the operation Φ^*_3 or Ψ^*_5 since G^*_{q+4} is the universal example for the operation Φ^*_3 or Ψ^*_5 . Obviously the conclusion of the theorem is satisfied for this choice of Φ^*_3 or Ψ^*_5 .

To prove the theorem for Φ^*_6 and Φ^*_5 the following consideration is needed:

Let $\hat{G}_{q+1}$ be the universal example for the operation Φ^*_3 or Φ^*_5 on (q+1)-dimensional integral cohomology class. Then it is easy to see that there exists a commutative diagram:

(3.4.13)

$$\begin{tikzcd}
 & & & & \hat{G}_{q+1} \arrow[dd, "\hat{p}_{q+1}"] & & \\
 & & & G^*_{q+1} \arrow[ur, "\bar{\rho}"] \arrow[dd, "\bar{p}_{q+1}"] & & & \\
 & \Sigma D^*_q \arrow[uurrr, "\hat{s}_1"] \arrow[rrr, dashed] & & & D^*_{q+1} \arrow[rr, "Sq^{q+1} d^*_{q+1}"] & & K_{2q+2} \\
\Sigma D^*_q \arrow[ur, equal] \arrow[uurrr, "s_1"] \arrow[rrr, "h"] & & & D^*_{q+1} \arrow[ur, equal] \arrow[rr, "\delta Sq^q d^*_{q+1}"] & & K^*_{2q+2} \arrow[ur, "\rho"] &
\end{tikzcd}$$

where $\rho : K^*_{2q+2} \longrightarrow K_{2q+2}$ is mod2reduction and $\bar{\rho} : G^*_{q+1} \rightarrow G_{q+1}$ is the induced map.

Let $v \in H^{2q+8}(\hat{G}_{q+4})$ be a representative for the operation Φ_6^* or Φ_5^* such that $\hat{i}^*_{q+4}(v) = Sq^4 \iota_{2q+4}$. Then

$$\hat{s}_1^* \sigma^3(v) = s_1^* \bar{\rho}^* \sigma^3(v) \quad \text{by commutativity of (3.4.13),}$$

$$= \gamma + h^* p^* \sigma^3(y) \text{ for some } y \in H^{2q+8}(K^*_{q+4})$$

by Proposition 3.4.10.

Define $\tilde{v} \in H^{2q+8}(G_{q+4})$ by $\tilde{v} = v - \hat{p}^*_{q+4} p^*(y)$. Thus

$$\hat{s}_1^* \sigma^3(\tilde{v}) = \hat{s}_1^* \sigma^3(v) - \hat{s}_1^* \sigma^3 \hat{p}^*_{q+4} p^*(y)$$

$$= \hat{s}_1^* \sigma^3(v) - \hat{s}_1^* \hat{p}^*_{q+1} p^* \sigma^3(y)$$

$$= \gamma .$$

Hence $s\, \hat{s}_1^* \sigma^3(\tilde{v}) = d_q^* \cup Sq^4 d_q^*$. Then it is easy to see that $\tilde{v} \in H^{2q+8}(\hat{G}_{q+4})$ is a candidate representing the stable secondary cohomology operation Φ_6^* or Φ_5^* satisfying the conclusion of Theorem 3.4.5. This completes the proof of Theorem 3.4.5 (1).

To prove part (2) of Theorem 3.4.5 we will need the following proposition which is an analogue of Proposition 3.4.11 when D_{q+1} is replaced by $\tilde{D}_{q+1}$.

3.4.14. PROPOSITION. Let $q \equiv 1$ or 2 (8) and let data be given by diagram (3.4.9). Suppose $\phi \in H^{2q+5}(\tilde{G}_{q+1})$ is such that

$$\tilde{i}^*_{q+1}(\phi) = Sq^4 \iota_{2q+1} .$$

Then $\quad s_1^*(\phi) = \gamma + h^*(x)$

where $\gamma \in H^{2q+5}(\Sigma \tilde{D}_q)$ is the unique class such that

$$s(\gamma) = e_q \cup Sq^4 e_q$$

and $x \in H^{2q+5}(\tilde{D}_{q+1})$ is given by

$$x = d + p^* \sigma^3(y) \text{ for some } y \in H^{2q+8}(K_{q+4}) ,$$

where $d \in H^{2q+5}(\tilde{D}_{q+1})$ is decomposable and $p:\tilde{D}_{q+1} \longrightarrow K_{q+1}$ is the principal fibration over K_{q+1} defining $\tilde{D}_{q+1}$.

Assuming that Proposition 3.4.14 is true we prove the second part of Theorem 3.4.5.

3.4.15. Proof of Theorem 3.4.5(2).

The existence of a stable secondary cohomology operation Φ_6 satisfying the conclusion of Theorem 3.4.5 follows from Proposition 3.4.11 in much the same way as the existence of Φ_6^* follows from Proposition 3.4.10 in 3.4.12. The existence of Φ_3 satisfying the conclusion of Theorem 3.4.5 follows from the existence of such stable secondary cohomology operation Φ_6 using a simple argument involving a diagram similar to Diagram (3.4.13) where G^*_{q+1}, D^*_{q+1}, ΣD^*_q and $\hat{G}_{q+1}$ are replaced by $\approx{G}_{q+1}$, D_{q+1}, ΣD_q and G_{q+1} respectively where $\approx{G}_{q+1}$ is the universal example for the operation Φ_3.

Similarly the existence of a stable secondary cohomology operation Φ_5 satisfying the conclusion of Theorem 3.4.5 follows from Proposition 3.4.14 in much the same way as the existence of such stable secondary cohomology operation Φ_6 follows from Proposition 3.4.11. The existence of a stable secondary cohomology operation Ψ_5 satisfying the conclusion of Theorem 3.4.5 follows from the existence of such stable secondary cohomology operation Φ_5 in the same way as the existence of such Φ_3 follows from the existence of such Φ_6.

This completes the proof of Theorem 3.4.5.

3.4.16. Proof of Proposition 3.4.10.

It is easily seen that for any cohomology class $\phi \in H^{2q+5}(G^*_{q+1})$ such that $i^*_{q+1}(\phi) = Sq^4 \iota^*_{2q+1}$

$$i^* s^*_1(\phi) = s^*_2 i^*_{q+1}(\phi) = Sq^4 d^*_q * d^*_q + d^*_q * Sq^4 d^*_q .$$

Now let $\gamma^* \in H^{2q+5}(\Sigma D^*_q)$ be the unique class such that

$$s(\gamma^*) = d^*_q \cup Sq^4 d^*_q .$$

Then $i^*(\gamma^*) = Sq^4 d^*_q * d^*_q + d^*_q * Sq^4 d^*_q$ by the commutativity of Diagram (3.1.4). Thus

$$i^*(s^*_1(\phi) - \gamma^*) = 0.$$

Therefore by the exactness of the Serre exact sequence for the fibration $h:\Sigma D^*_q \longrightarrow D^*_{q+1}$

$$s^*_1(\phi) = \gamma^* + h^*(x) \text{ for some } x \in H^{2q+5}(D^*_{q+1}) .$$

Let $\tilde{i}:K_{q+2} \longrightarrow D^*_{q+1}$ be the inclusion of the fibre into the total space of the fibration $p:D^*_{q+1} \longrightarrow K^*_{q+1}$ induced by $Sq^2 \iota^*_{q+1}$.

Then
$$\begin{aligned} 0 = \tilde{i}^* s\, s^*_1(\phi) &= \tilde{i}^* s(\gamma^*) + \tilde{i}^* s\, h^*(x) \\ &= \tilde{i}^* \sigma(x) \\ &= \sigma \tilde{i}^*(x) . \end{aligned}$$

But $\operatorname{Ker}(\sigma:H^{2q+5}(K_{q+2}) \longrightarrow H^{2q+4}(K_{q+1})) = \langle \iota_{q+2} \cdot Sq^1 \iota_{q+2} \rangle_{Z_2}$

Thus $\tilde{i}^*(x) = \alpha \cdot \iota_{q+2} \cdot Sq^1 \iota_{q+2}$ for some $\alpha \in Z_2$. We claim that $\alpha = 0$. Because otherwise if $\alpha \neq 0$ then

$$j^*(\iota_{q+2} \cdot Sq^1 \iota_{q+2}) = 0$$

where $\tilde{i}:K_{q+2} \longrightarrow D^*_{q+1}$ is considered as a fibration and $j:K^*_q \longrightarrow K_{q+2}$ is the inclusion of its fibre into the total space. This then contradicts the fact that $j^*(\iota_{q+2} \cdot Sq^1 \iota_{q+2}) = Sq^2 \iota^*_q \cdot Sq^3 \iota^*_q \neq 0$.

Thus we may assume that $x \in \operatorname{Ker}(\tilde{i}^*)$. Now Lemma 3.1.13 gives the following exact sequence:

$$(3.4.17)\quad H^{2q+5}(K^*_{q+1}) \xrightarrow{p^*} H^{2q+5}(D^*_{q+1}) \cap \operatorname{Ker}(\tilde{i}^*) \xrightarrow{\mu}$$
$$\xrightarrow{\mu} H^{2q+5}(K_{q+2} \wedge D^*_{q+1}) \xrightarrow{\tau} H^{2q+6}(K^*_{q+1}) \,/\, (Sq^2 \iota^*_{q+1})^2$$

Now that $H^{2q+5}(K_{q+2} \wedge D^*_{q+1}) \cong \langle Sq^2 \iota_{q+2} \otimes d^*_{q+1} \rangle_{Z_2}$ and that

$$\mu(\psi_{q+4} \cdot d^*_{q+1}) = Sq^2 \iota_{q+2} \otimes d^*_{q+1} \quad \text{by Lemma 3.1.11}$$

where ψ_{q+4} is the unique class such that $\tilde{i}^*(\psi_{q+4}) = Sq^2 \iota_{q+2}$; from which we deduce by the exactness of (3.4.17) that

$$\mu(x - \beta\, d^*_{q+1} \cdot \psi_{q+4}) = 0 \quad \text{for some } \beta \in Z_2 .$$

Therefore $\quad x = \beta\, d^*_{q+1} \cdot \psi_{q+4} + p^*(\tilde{x})$ for some $\tilde{x} \in H^{2q+5}(K^*_{q+1})$. It is easily seen that there exists a class $y \in H^{2q+8}(K^*_{q+4})$ such that $\quad p^*(\tilde{x}) = p^* \sigma^3(y)$. Thus we have shown that

$x = d + p^* \sigma^3(y) \quad$ where $d \in H^{2q+5}(D^*_{q+1})$ is decomposable and $s_1^*(\phi) = \gamma^* + h^*(x)$. This completes the proof of Proposition 3.4.10.

3.4.18. Proof of Proposition 3.4.11.

Similar to the proof of Proposition 3.4.10 we find that for any cohomology class $\phi \in H^{2q+5}(G_{q+1})$ such that

$$i^*_{q+1}(\phi) = Sq^4 \iota_{2q+1} ,$$

there exists a cohomology class $x \in H^{2q+5}(D_{q+1})$ such that

$$s_1^*(\phi) = \gamma + h^*(x) .$$

As in 3.4.16, $\quad 0 = \tilde{i}^*\, s\, s_1^*(\phi) = \tilde{i}^*\, s(\gamma) + \tilde{i}^*\, s\, h^*(x)$

$$= \tilde{i}^* \sigma(x) = \sigma\, \tilde{i}^*(x) ,$$

where $\tilde{i}:K_{q+2} \longrightarrow D_{q+1}$ is the inclusion of the fibre in the total space of the fibration $p:D_{q+1} \longrightarrow Y_{q+1}$ induced by $Sq^2 b_{q+1}$ (where b_{q+1} is the fundamental class of Y_{q+1}). Thus

$$\tilde{i}^*(x) = \alpha\, \iota_{q+2} . Sq^1 \iota_{q+2} \quad \text{for some } \alpha \in Z_2 .$$

We claim that $\alpha = 0$. Otherwise suppose $\alpha \neq 0$ then

$$j^*(\iota_{q+2} . Sq^1 \iota_{q+2}) = 0$$

where $j:Y_q \longrightarrow K_{q+2}$ is the inclusion of the fibre in the total space of $\tilde{i}:K_{q+2} \longrightarrow D_{q+1}$ when it is regarded as a fibration. This then contradicts the fact that

$$j^*(\iota_{q+2} . Sq^1 \iota_{q+2}) = Sq^2 b_q . Sq^3 b_q \neq 0.$$

Hence $\tilde{i}^*(x) = 0$. Now Lemma 3.1.13 gives the following exact sequence:

$$(3.4.19) \qquad H^{2q+5}(Y_{q+1}) \xrightarrow{p^*} H^{2q+5}(D_{q+1}) \cap \mathrm{Ker}(\tilde{i}^*) \longrightarrow$$

$$\xrightarrow{\mu} H^{2q+5}(K_{q+2} \wedge D_{q+1}) \xrightarrow{\tau} H^{2q+6}(Y_{q+1}) \,/\, (Sq^2 b_{q+1})^2 .$$

Now $H^{2q+5}(K_{q+2} \wedge D_{q+1}) \cong \langle Sq^1 \iota_{q+2} \otimes \psi_{q+2} \,,\, Sq^2 \iota_{q+2} \otimes d_{q+1} \rangle_{Z_2}$ where $\psi_{q+2} \in H^{q+2}(D_{q+1})$ is equal to $p^*(\zeta_{q+2})$ where $\zeta_{q+2} \in H^{q+2}(Y_{q+2})$ is the unique class representing the stable cohomology operation defined by the relation in $\mathcal{A}(2)$, $Sq^1 Sq^1 = 0$. But $\tau(Sq^1 \iota_{q+2} \otimes \psi_{q+2}) = Sq^3 b_{q+1} . \zeta_{q+2} \neq 0$ whereas $\tau(Sq^2 \iota_{q+1} \otimes d_{q+1}) = 0$. Hence $Sq^1 \iota_{q+2} \otimes \psi_{q+2}$ is not in the image of μ. Now let $\tilde{\psi}_{q+4} \in H^{q+4}(D_{q+1})$ be the unique class such that $\tilde{i}^*(\tilde{\psi}_{q+4}) = Sq^2 \iota_{q+2}$; then $\mu(\tilde{\psi}_{q+4} . d_{q+1}) = Sq^2 \iota_{q+2} \otimes d_{q+1}$ by Lemma 3.1.11. Thus by the exactness of the sequence (3.4.19),

$$x = \beta\, d^*_{q+1} . \tilde{\psi}_{q+4} + p^*(\tilde{x}) \quad \text{for some } \beta \in Z_2 \text{ and}$$

$\tilde{x} \in H^{2q+5}(Y_{q+1})$. From this it is trivial to deduce that there exists a cohomology class $y \in H^{2q+8}(Y_{q+4})$ such that

$$x = \text{decomposable} + p^* \sigma^3(y) .$$

This completes the proof of Proposition 3.4.11.

3.4.20. Proof of Proposition 3.4.14. The proof is similar to that of Proposition 3.4.11 in 3.4.18 and is omitted.

§ 3.5. Application to the Identification of k-invariants.

3.5.1. THEOREM. Notation as given by Diagram (2.1.4). There exist stable secondary cohomology operations Φ_i^* i = 1, 2,...,6 , such that

$$\Phi_i^*((Tq_1)^*(U_{BSpin_n})) = \{ (Tq_1)^*(U_{BSpin_n}).k_i^2 \}$$

for i = 1, 2, 4, 5 and 6 ;

$$\Phi_3^*((Tq_1)^*(U_{BSpin_n})) = \{ (Tq_1)^*(U_{BSpin_n}).(k_3^2 + w_4.w_{n-7}) \}.$$

Proof. By Theorems 3.2.3, 3.3.3 and 3.4.5 there exist stable secondary cohomology operations Φ_i^* i = 1,2,4, 5,6 such that

$$\Phi_i^*((T\pi)^*(U_{BSpin_n})) = \{ 0 \} ,$$

Therefore the assertion follows from Theorem 2.2.17. We are left with proving the assertion for i = 3. By Theorem 3.4.5(1) there exists stable secondary cohomology operation Φ_3^* such that $\quad s^7\Phi_3^*((T\pi)^*(U_{BSpin_n})) \ni (T\pi)^*(U_{BSpin_n}).(w_4.w_{n-7})$.

Therefore by Lemma 3.1.7

$$(T\pi)^*(U_{BSpin_n}).(w_4.w_{n-7}) \in \Phi_3^*((T\pi)^*(U_{BSpin_n})).$$

The assertion then follows from Theorem 2.2.17. This completes the proof of Theorem 3.5.1.

3.5.2. Remark. One way to use the Admissible Class Theorem to identify the n-dimensional k-invariant for the third stage of the n-MPT for the fibration (2.1.2) is to alter the fibration (2.1.2) such that $w_4 = 0$ in the base. This follows from Table 1.2.16, Theorem 2.2.14 and Theorem 3.5.1 since we can find a tertiary operation satisfying the hypothesis of Theorem 2.2.14, which will identify the new k-invariant and hence will give some control over the original k-invariant. This will be carried out in the next chapter.

CHAPTER 4. A TERTIARY COHOMOLOGY OPERATION AND THE SINGLE OBSTRUCTION TO LIFTING

This chapter is concerned mainly with the identification of the n-dimensional component of the k-invariant for the third stage of the n-MPT for the fibration,

$$V_{n,7} \longrightarrow BSO_{n-7}[8] \longrightarrow BSO_n[8] \quad ,$$

induced canonically from the fibration (2.1.2), where $X[j]$ denotes the (j-1)-connective covering over X for any based space X.

§4.1. The chain complex $\mathcal{C}_n$ for $n \equiv 7\ (8) \geqslant 15$

We assume familiarity with C.R.F. Maunder [21] . Since we are concerned with a particular tertiary cohomology operation, its definition is self evident in the definition of the universal example for a tertiary cohomology operation; moreover, in the next chapter can be found an axiomatised definition of an N^{th} order cohomology operation due to F. Adams and C.R.F. Maunder. We leave the exploration into some of the properties of N^{th} order cohomology operations, in particular, the relation of an operation to its (S-) dual operation, to the next chapter.

4.1.1. The complex $\mathcal{C}_n$ ($n \equiv 7\ (8) \geqslant 15$)

Let

$$(4.1.2)\quad \mathcal{C}_n :\ C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3$$

be the chain complex of $\mathcal{A}(2)$-modules and homomorphisms defined as follows:

C_0 is free on one generator c_0 of degree 0;

C_1 is free on $\{ c_{1,1}, c_{1,2}, c_{1,n-6}, c_{1,n-5}, c_{1,n-3} \}$ where degree of $c_{1,i} = i$;

C_2 is free on $\{ c_{2,n-4}, c_{2,n-3}, c^1_{2,n-2}, c^2_{2,n-2}, c_{2,n-1}, c_{2,n+1}, c_{2,4} \}$ where degree of $c_{2,i} = i$;

C_3 is free on one generator $c_{3,n+2}$ of degree $n+2$;

and the differentials are defined below:

$$d_1c_{1,1} = Sq^1c_0\ ;\ d_1c_{1,2} = Sq^2c_0;$$

$$d_1c_{1,n-3} = Sq^{n-3}c_0;\ d_1c_{1,n-5} = Sq^{n-5}c_0;\ d_1c_{1,n-6} = Sq^{n-6}c_0;$$

$$d_2c_{2,4} = Sq^3c_{1,1} + Sq^2c_{1,2}\ ;$$

$$d_2c_{2,n-4} = Sq^{n-5}c_{1,1} + Sq^2c_{1,n-6};$$

$$d_2c_{2,n-3} = Sq^{n-4}c_{1,1} + Sq^2c_{1,n-5};$$

$$d_2c^1_{2,n-2} = Sq^{n-3}c_{1,1} + Sq^{n-4}c_{1,2} + Sq^4c_{1,n-6};$$

$$d_2c^2_{2,n-2} = Sq^{n-3}c_{1,1} + Sq^1c_{1,n-3} + Sq^2Sq^1c_{1,n-5};$$

$$d_2c_{2,n-1} = Sq^{n-3}c_{1,2} + Sq^4c_{1,n-5};$$

$$d_2c_{2,n+1} = Sq^nc_{1,1} + Sq^{n-1}c_{1,2} + Sq^4c_{1,n-3};$$

and

$d_3: C_3 \longrightarrow C_2$ is given by

$$d_3c_{3,n+2} = (Sq^2Sq^4)c_{2,n-4} + (Sq^4 + Sq^3Sq^1)c_{2,n-2} + Sq^{n-2}c_{2,4}\ .$$

4.1.3. LEMMA. $\mathcal{C}_n$ is admissible in the sense of C.R.F. Maunder. (i.e. $\mathcal{C}_n$ admits a geometric realisation.)

Proof. First we show that there is a geometrical realisation for the chain complex

(4.1.4) $$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \quad .$$

This is easily seen for this chain complex is associated with the (vector) cohomology operation $(\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \psi_1)$ where Φ_i for i = 1,2,.., 6 are the stable secondary cohomology operations given in chapter 3 and ψ_1 is the stable secondary cohomology operation associated with the relation in $\mathcal{A}(2)$

(4.1.5) $$Sq^2Sq^2 + Sq^3Sq^1 = 0 \ .$$

Let $p_1 : P_1 \longrightarrow K_n$ be the principal fibration with classifying map $f_1 = (Sq^{n-6}\iota_n, Sq^{n-5}\iota_n, Sq^{n-3}\iota_n, Sq^2\iota_n, Sq^1\iota_n) : K_n \longrightarrow M_1$ where $M_1 = K_{2n-6} \times K_{2n-5} \times K_{2n-3} \times K_{n+2} \times K_{n+1}$. Then we have the following commutative diagram:

(4.1.6) $$\begin{array}{ccccc} T^*(P_1) & \xrightarrow{\tau} & H^{*+1}(M_1) & \xrightarrow{f_1^*} & H^{*+1}(K_n) \\ \uparrow f_2 & & \uparrow \beta_1 & & \uparrow \beta_0 \\ C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 \end{array}$$

where $\beta_0 : (C_0)_t \longrightarrow H^{n+t}(K_n)$ is an isomorphism for $t < n$, $\beta_1 : (C_1)_t \longrightarrow H^{n+t}(M_1)$ is an isomorphism for $t < (n+2)$ and $f_2 : C_2 \longrightarrow T^*(P_1) \rightarrow H^*(P_1)$ is the homomorphism that maps the generators to their corresponding representatives for the stable secondary cohomology operations, and $T^*(P_1)$ is defined in Definition 3.1.8. Note that $T^*(P_1)$ is isomorphic to $H^*(P_1)$

in dimensions less than or equal to $(2n-1)$ and it embeds in $H^*(P_1)$ monomorphically in dimensions less than or equal to $2n$. Corresponding to this choice of operations is the principal fibrations, $p_2 : P_2 \longrightarrow P_1$,

with classifying map $\tilde{f}_2 = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \psi_1) : P_1 \longrightarrow M_2$ where ϕ_i $i = 1, 2, \ldots, 5, 6$ and ψ_1 are the representatives for the operations and $M_2 = K_{2n-5} \times K_{2n-4} \times K^1_{2n-3} \times K^2_{2n-3} \times K_{2n-2} \times K_{2n} \times K_{n+3}$. Thus we obtain the following diagram:

(4.1.7)
$$\begin{tikzcd}
 & H^*(P_1) & \\
 & T^*(P_1) \arrow[u] \arrow[r, "\tau"] & H^{*+1}(M_1) \\
C_3 \arrow[r, "d_3"] \arrow[d, dashed, "f_3"] & C_2 \arrow[u, "f_2"] \arrow[r, "d_2"] \arrow[d, "\beta_2"] & C_1 \arrow[u, "\beta_1"] \\
H^{*-1}(P_2) \arrow[r, "\tau"] & H^*(M_2) \arrow[uuu, bend left, "\tilde{f}_2^*"] &
\end{tikzcd}$$

where $\tilde{f}_2^* \circ \beta_2 = f_2$, $\beta_1 \circ d_2 = \tau \circ f_2$ and $\beta_2 : (C_2)_t \longrightarrow H^{n+t-1}(M_2)$ is an isomorphism for $t \leq (n+7)$.

Our aim is to show that $\beta_2 \circ d_3$ factors through $H^*(P_2)$; i.e., we want to show that $f_2 \circ d_3$ is the zero homomorphism, because the image of the generator $c_{3,n+2}$ under $f_2 \circ d_3$ lies in the range of dimensions within the range of the Serre exact sequence for the fibration, $P_2 \longrightarrow P_1 \longrightarrow M_2$. Thus Lemma 4.1.3 would be proved if we had shown that

(4.1.8)
$$Sq^2 Sq^4 \phi_1 + (Sq^4 + Sq^3 Sq^1)\phi_3 + Sq^{n-2}\psi_1 = 0 .$$

4.1.9. LEMMA. For the choice of stable secondary cohomology operations above, (4.1.8) is satisfied.

In order to prove Lemma 4.1.9 we need the following proposition:

4.1.10. PROPOSITION. Notation as in Definition 3.1.8.

(a) $\Lambda^{2n+2} = \{ Sq^1\iota_n \cup Sq^1\iota_n \}$ and

(b) the following sequence is exact:

$$H^{2n+1}(K_n) \longrightarrow H^{2n+1}(P_1) \cap \operatorname{Ker}(i_1^*) \xrightarrow{\mu} T^{2n+1}(\Omega M_1 \wedge P_1)$$

where $i_1 : \Omega M_1 \to P_1$ is the inclusion of the fibre in the total space of the fibration $p_1 : P_1 \to K_n$.

Assuming that Proposition 4.1.10 is true we shall prove Lemma 4.1.9 and hence Lemma 4.1.3.

4.1.11. Proof of Lemma 4.1.9.

Let $\mu: H^*(P_1) \longrightarrow H^*(\Omega M_1 \times P_1,\ P_1)$ be the homomorphism defined in 1.3.2. Then by choice

$$\mu(\phi_1) = Sq^2\iota_{2n-7}\otimes 1 + Sq^{n-5}\iota_n \otimes 1 \,,$$

$$\mu(\phi_3) = Sq^4\iota_{2n-7}\otimes 1 + Sq^{n-4}\iota_{n+1}\otimes 1 + Sq^{n-3}\iota_n\otimes 1 \quad \text{and}$$

$$\mu(\psi_1) = Sq^2\iota_{n+1}\otimes 1 + Sq^3\iota_n \otimes 1 \,.$$

Thus, $i_1^*(Sq^2Sq^4\phi_1 + (Sq^4+Sq^3Sq^1)\phi_3 + Sq^{n-2}\psi_1) = 0$. Also

$$\mu\,(Sq^2Sq^4\phi_1 + (Sq^4+Sq^3Sq^1)\phi_3 + Sq^{n-2}\psi_1) = 0$$

where $\mu: H^*(P_1) \longrightarrow H^*(\Omega M_1 \wedge P_1)$ is the obvious homomorphism arising from the homomorphism μ above. By abuse of notation we have denoted them by the same letter.

Therefore by Lemma 4.1.10(b) there is a class $a \in H^{2n+1}(K_n)$ such that $Sq^2Sq^4\phi_1 + (Sq^4+Sq^3Sq^1)\phi_3 + Sq^{n-2}\psi_1 = p_1^*(a)$.

But by Corollary 3.2.4, Theorem 3.4.5(2) and the fact that $\sigma^7(Sq^{n-2}\psi_1) = 0$,

$$\sigma^7(p_1^*(a)) = \sigma^7(Sq^2Sq^4\phi_1 + (Sq^4 + Sq^3Sq^1)\phi_3 + Sq^{n-2}\psi_1) = 0 .$$

Thus this shows that $\sigma^7(a)$ belongs to $\mathrm{Ker}((\Omega^7 p_1)^*)$. Now $\mathrm{Ker}((\Omega^7 p_1)^*)$ is the left ideal generated by $\{Sq^1\iota_{n-7}, Sq^2\iota_{n-7}\}$; also from the cohomology structure of K_n and that of K_{n-7} one easily deduces that

$$\mathrm{Ker}(\sigma^7) \in \mathrm{Ker}(p_1^*) \cap H^{2n+1}(K_n).$$

This is enough to tell us that $p_1^*(a)$ is zero. For on the contrary suppose that $p_1^*(a) \neq 0$ then this implies that $\sigma^7(a) \neq 0$. But $(\Omega^7 p_1)^*(\sigma^7(a)) = 0$ and $\sigma^7(a) \neq 0$ would imply by the above analysis that $p_1^*(a) = 0$. This is absurd; and thus $p_1^*(a)$ must necessary be zero.

This completes the proof of Lemma 4.1.9 modulo the proof of Proposition 4.1.10.

4.1.12. Proof of Proposition 4.1.10.

Proof of part (a). Let $\overline{p}_1 : (P_1, \Omega M_1 \times P_1) \longrightarrow (K_n, P_1)$ be the map of pair as given in Definition 3.1.8 when the fibration $p: E \longrightarrow B$ there is replaced by $p_1 : P_1 \rightarrow K_n$. Let $j_1 : K_n \rightarrow (K_n, P_1)$ be the inclusion map. Consider the following commutative diagram:

$$\begin{array}{lccccccc}
 & H^{*-1}(P_1) & \xrightarrow{m^*} & H^{*-1}(\Omega M_1 \times P_1) & \xrightarrow{\delta_2} & H^*(P_1, \Omega M_1 \times P_1) & \rightarrow & H^*(P_1) \\
(4.1.13) & & & & & \uparrow{\scriptstyle \overline{p}_1^*} & & \uparrow{\scriptstyle p_1^*} \\
 & & & & & H^*(K_n, P_1) & \xrightarrow{j_1^*} & H^*(K_n)
\end{array}$$

where $m:\Omega M_1 \times P_1 \longrightarrow P_1$ is the action of the fibre on the total space of the principal fibration. Note that by a Leray-Serre spectral sequence argument $\overline{p}_1^*$ is an isomorphism in dimensions less than or equal to 2n and is a monomorphism in dimensions less than or equal to (2n+1). By exactness of the cohomology sequence for the pair (K_n, P_1) there exist cohomology classes $a_1, a_2, a_3, a_4, a_5 \in H^*(K_n, P_1)$ such that

$$j_1^*(a_1) = Sq^1 \iota_n ,$$
$$j_1^*(a_2) = Sq^2 \iota_n ,$$
$$j_1^*(a_3) = Sq^{n-6} \iota_n ,$$
$$j_1^*(a_4) = Sq^{n-5} \iota_n \text{ and}$$
$$j_1^*(a_5) = Sq^{n-3} \iota_n .$$

Now for $t < 2n$ we have the following commutative diagram:

(4.1.14)
$$\begin{tikzcd}
 & H^t(\Omega M_1 \times P_1) \arrow[r] \arrow[drr, "\tau_1"] & H^t(\Omega M_1) \arrow[dr, "\tau"] & & \\
H^t(P_1) \arrow[r, "\mu"] & H^t(\Omega M_1 \times P_1, P_1) \arrow[u, "j_2^*"] \arrow[rr, "\tau"] & & H^{t+1}(K_n) \arrow[r, "p_1^*"] & H^{t+1}(P_1)
\end{tikzcd}$$

where $\tau : H^t(\Omega M_1) \longrightarrow H^{t+1}(K_n)$ is the transgression in the fibration $p_1 : P_1 \longrightarrow K_n$, the lower row is exact (see Thomas [36]). It follows from the definition of transgression and the commutativity of diagram (4.1.14) that the classes $\{a_i\}_{1<i<5}$ can be chosen such that

$$(4.1.15)\begin{cases} \delta_2(\iota_n \otimes 1) = \overline{p}_1^*(a_1), \quad \delta_2(\iota_{n+1} \otimes 1) = \overline{p}_1^*(a_2), \\ \delta_2(\iota_{2n-7} \otimes 1) = \overline{p}_1^*(a_3), \quad \delta_2(\iota_{2n-6} \otimes 1) = \overline{p}_1^*(a_4) \text{ and} \\ \delta_2(\ {}_{2n-4} \otimes 1) = \overline{p}_1^*(a_5) . \end{cases}$$

Notice that $\delta_2: H^*(\Omega M_1 \times P_1) \to H^{*+1}(P_1, \Omega M_1 \times P_1)$ has the property that it is a $H^*(K_n)$-homomorphism (via p_1^*). Recall (Thomas [36]) that the following composition

$$\begin{tikzcd} H^*(\Omega M_1 \times P_1, P_1) \arrow[rr, "h^*"] \arrow[dr, "j_2^*"'] & & H^*(P_1, \Omega M_1 \times P_1) \\ & H^*(\Omega M_1 \times P_1) \arrow[ur, "\delta_2"'] & \end{tikzcd}$$

is an isomorphism. This implies that for $i \in \{n, n+1, 2n-7, 2n-6, 2n-4\}$

$$\delta_2(Sq^I(\iota_i \otimes 1)) \neq 0 \text{ unless } Sq^I(\iota_i \otimes 1) = 0$$

for any admissible sequence I. Thus $Sq^I \overline{p}_1^*(a_i) \neq 0$ for any admissible sequence with excess less than or equal to n and for $i \in \{1,2,3,4,5\}$. Therefore in dimension (2n+1)

$$\begin{aligned} \overline{p}_1^*(a_2 \cup \iota_n) &= \overline{p}_1^*(a_2) \cup p_1^*(\iota_n) \\ &= \delta_2(\iota_{n+1} \otimes 1) \cup p_1^*(\iota_n) \\ &= \delta_2(\iota_{n+1} \otimes p_1^*(\iota_n)) \\ &\neq 0 . \quad \text{because } h^* \text{ is an isomorphism.} \end{aligned}$$

Also $\overline{p}_1^*(Sq^{n+1} a_1) = 0$ because $Sq^{n+1}\delta_2(\iota_n \otimes 1) = 0$. Thus

$$Sq^{n+1} a_1 \in \operatorname{Ker}(\overline{p}_1^*) ;$$

and since this is the only class belonging to $(\operatorname{Ker}(\overline{p}_1^*))_{2n+2}$ modulo $\operatorname{Ker}(j_1^*)$

$$\begin{aligned} \Lambda^{2n+2} &= (j_1^*(\operatorname{Ker}(\overline{p}_1^*)))_{2n+2} = \{ Sq^{n+1} j_1^*(a_1) \} \\ &= \{ Sq^1 \iota_n \cup Sq^1 \iota_n \}. \end{aligned}$$

This proves part (a) of Proposition 4.1.10.

Proof of Proposition 4.1.10 part (b).

Suppose $y \in \operatorname{Ker}(\mu) \cap \operatorname{Ker}(i_1^*) \cap H^{2n+1}(P_1)$. Then this implies that $\bar{p}_1^*(\delta_1(y)) = 0$ where $\delta_1 : H^*(P_1) \longrightarrow H^{*+1}(K_n, P_1)$ is the coboundary homomorphism for the pair (K_n, P_1). By part (a)

$$\delta_1(y) = \alpha \cdot a_1^2 \quad \text{for some } \alpha \in Z_2 .$$

But by exactness of the long exact sequence for the pair (K_n, P_1)

$$j_1^*(\delta_1(y)) = 0.$$

Thus $\quad \alpha \cdot Sq^1 \iota_n \cup Sq^1 \iota_n = j_1^*(\delta_1(y)) = 0$ implies that $\alpha = 0$.

Hence $\quad \delta_1(y) = 0$. By exactness of the cohomology sequence for the pair (K_n, P_1), $y \in \operatorname{Im}(p_1^*)$.

On the other hand it is trivial that

$$\mu \circ p_1^* = 0 . \quad \text{Thus } \operatorname{Im}(p_1^*) = \operatorname{Ker}(\mu) .$$

This proves part (b) and the proof of Proposition 4.1.10 is completed.

§ 4.2. The Operation Associated with the Chain Complex $\mathcal{C}_n$.

Lemma 4.1.3 shows that one can define a family of tertiary cohomology operations associated with the chain complex $\mathcal{C}_n$ of 4.1.1. More specifically the family of tertiary operations is associated with the relation (4.1.8) among the stable secondary cohomology operations Φ_1, Φ_3 and Ψ_1. On mod two cohomology classes of dimension (n-7) in the domain of Φ_1, Φ_3 and Ψ_1 the relation (4.1.8) becomes

(4.2.1) $$Sq^2 Sq^4 \Phi_1 + (Sq^4 + Sq^3 Sq^1)\Phi_3 = 0.$$

For $k > 4$ let D_k be the universal example for k-dimensional mod 2 cohomology class x satisfying $Sq^1 x = Sq^2 x = Sq^4 x = 0$. Let $d_k \in H^k(D_k)$ be the fundamental class of D_k. Let g_1, g_2, g_3 $g_4 \in H^*(D_k)$ be the unique classes representing the stable basic secondary cohomology operations associated with the relations:

(4.2.2) $$Sq^1 Sq^1 = 0,$$

(4.2.3) $$Sq^2 Sq^2 + Sq^3 Sq^1 = 0,$$

(4.2.4) $$(Sq^2 Sq^1) Sq^2 + Sq^4 Sq^1 + Sq^1 Sq^4 = 0 \text{ and}$$

(4.2.5) $$Sq^4 Sq^4 + Sq^6 Sq^2 + Sq^7 Sq^1 = 0.$$

Note that $g_2 \in H^{k+3}(D_k)$ represents the operation Ψ_1 on k-dimensional cohomology classes in its domain. Let $r: Y_k \to D_k$ be the principal fibration over D_k with g_2 as classifying map. Then Y_k is the universal example for k-dimensional mod 2 cohomology class x satisfying $Sq^1 x = Sq^2 x = Sq^4 x = 0$ and $\Psi_1(x) = 0$.

Observe the following easily proven facts:

4.2.6. FACTS.

(1) $r^*: H^i(D_{n-7}) \longrightarrow H^i(Y_{n-7})$ is an epimorphism for $i \leq (n-1)$

(2) Let $h_1 = r^*(g_1)$, $h_3 = r^*(g_3)$ and $h_4 = r^*(g_4)$. In dimensions $\leq n$ the following relations hold in $H^*(Y_{n-7})$

$$Sq^1 h_1 = 0 \text{ and}$$

$$Sq^1 h_3 + Sq^4 h_1 = 0.$$

Moreover these are the basic relation in dimensions $\leq n$.

(3) Let $\tilde{\iota}: K_{n-5} \longrightarrow Y_{n-7}$ be the fibre inclusion. Then there exists a class $u \in H^n(Y_{n-7})$ such that $\tilde{\iota}^*(u) = Sq^2 Sq^3 \iota_{n-5}$.

In fact u represents a tertiary cohomology operation associated with the relation

(4.2.7) $\quad Sq^2 Sq^3 \Psi_1 = 0$.

Let $b_k \in H^k(Y_k)$ be the fundamental class of Y_k . Then the following lemma is easily proven :

4.2.8. LEMMA.

$H^{n-7}(Y_{n-7}) \cong \langle b_{n-7} \rangle_{Z_2}$;

$H^{n-6}(Y_{n-7}) \cong \langle h_1 \rangle_{Z_2}$; $H^{n-5}(Y_{n-7}) \cong 0$;

$H^{n-4}(Y_{n-7}) \cong \langle Sq^2 h_1 \rangle_{Z_2}$;

$H^{n-3}(Y_{n-7}) \cong \langle Sq^3 h_1, h_3 \rangle_{Z_2}$; $H^{n-2}(Y_{n-7}) \cong \langle Sq^4 h_1 \rangle_{Z_2}$;

$H^{n-1}(Y_{n-7}) \cong \langle Sq^2 h_3 \rangle_{Z_2}$;

$H^n(Y_{n-7}) \cong \langle Sq^6 h_1, Sq^4 Sq^2 h_1, Sq^3 h_3, h_4, u \rangle_{Z_2}$.

The main theorem of this section is the following:

4.2.9. THEOREM. There exists a stable tertiary cohomology operation Θ associated with the chain complex $\mathcal{C}_n$ of (4.1.1) such that $b_{n-7} \cup (h_4 + Sq^3 h_3) \in \Theta(b_{n-7})$.

Proof. It is easily seen that the following diagram is commutative:

(4.2.10)
$$\begin{tikzcd}
\Omega M_2 \arrow[d] \arrow[rr, equal] & & \Omega M_2 \arrow[d] \arrow[rr, equal] & & \Omega M_2 \arrow[d, "i_{2,n-6}"] & \\
\widetilde{G}_2 \times K_{2n-11} \times K^2_{2n-10} \times K_{2n-9} \times K_{2n-7} \arrow[d] \arrow[rr, "j_2"] \arrow[rr, phantom, shift right=4ex, "1"] & & G_2 \arrow[d] \arrow[rr, "\zeta"] \arrow[rr, phantom, shift right=4ex, "2"] & & P_{2,n-6} \arrow[d, "p_{2,n-6}"] \arrow[r, "\theta"] & K_{2n-6} \\
K_{2n-13} \arrow[rr, "j_1"] & & \widetilde{\Omega M}_1 \arrow[rr, "i_{1,n-6}"] & & P_{1,n-6} \arrow[d, "p_{1,n-6}"] \arrow[r] & \widetilde{M}_2 \\
& Y_{n-7} * Y_{n-7} \arrow[ul, "s_3"] \arrow[uul, "\xi_3"'] \arrow[uur, "\xi_2"] \arrow[ur, "s_2"'] \arrow[rr, "i"'] & & \Sigma Y_{n-7} \arrow[ur, "s_1"] \arrow[uur, "\xi_1"] \arrow[r, "h"'] & Y_{n-6} \arrow[r] & \widetilde{M}_1
\end{tikzcd}$$

where $\widetilde{M}_1 = K_{2n-12} \times K_{2n-11} \times K_{2n-9}$,

$\widetilde{M}_2 = K_{2n-11} \times K_{2n-10} \times K^1_{2n-9} \times K^2_{2n-9} \times K_{2n-8} \times K_{2n-6}$,

$p_{1,k}$ is the principal fibration over Y_k with classifying map $(Sq^{n-6} b_k, Sq^{n-5} b_k, Sq^{n-3} b_k)$,

$\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \in H^*(P_{1,n-6})$ are the cohomology classes representing the stable secondary cohomology operations given by Chapter 3,

$p_{2,n-6} : P_{2,n-6} \to P_{1,n-6}$ is the principal fibration with classifying map $(\phi_1, \phi_2, \ldots, \phi_6)$,

$\widetilde{G}_2$ is the universal example for $(2n-13)$-dimensional mod 2 cohomology class x satisfying $Sq^2 x = Sq^4 x = 0$,

i, $i_{1,n-6}$, $i_{2,n-6}$ are the respective fibre inclusions,

$j_1: K_{2n-13} \longrightarrow M_1$ is the obvious inclusion,

'squares' (1) and (2) are fibre squares,

s_1 is a lifting such that $s_1^*(\phi_i) = 0$ for $i = 1, 2, \ldots, 6$,

s_2 is the restriction of s_1, s_3 represents $b_{n-7}*b_{n-7}$,

ξ_1 is a lifting of s_1 to $P_{2,n-6}$,

ξ_2, ξ_3 are the maps given by the fibre square property.

That $s_2: Y_{n-7}*Y_{n-7} \longrightarrow M_1$ factors through s_3 is seen as follows:

The following cohomology groups of $Y_{n-7}*Y_{n-7}$ are specified by their generators:

$$H^{2n-13}(Y_{n-7}*Y_{n-7}) \cong \langle b_{n-7}*b_{n-7} \rangle_{Z_2} ,$$

$$H^{2n-12}(Y_{n-7}*Y_{n-7}) \cong \langle b_{n-7}*h_1 , h_1*b_{n-7} \rangle_{Z_2} ,$$

$$H^{2n-10}(Y_{n-7}*Y_{n-7}) \cong \langle b_{n-7}*Sq^2h_1 , Sq^2h_1*b_{n-7} \rangle_{Z_2} .$$

Now $s_2^*(\iota_{2n-12}) \neq b_{n-7}*h_1$ because by our choice of s_2 $s_2^*(Sq^2\iota_{2n-12}) = 0$ whereas $Sq^2(b_{n-7}*h_1) \neq 0$. Similarly $s_2^*(\iota_{2n-10}) \neq b_{n-7}*Sq^2h_1$ because $Sq^4(b_{n-7}*Sq^2h_1) \neq 0$. This is enough to deduce that $s_2^*(\iota_{2n-12}) = s_2^*(\iota_{2n-10}) = 0$. Since $\tau(s_2^*(\iota_{2n-13})) = b_{n-7} \cup b_{n-7}$, $s_2^*(\iota_{2n-13})$ must equal $b_{n-7}*b_{n-7}$. Thus such s_2 factors through s_3 which represents the fundamental class of $Y_{n-7}*Y_{n-7}$.

It follows from the relation (4.1.3) and the Serre exact sequence for the fibration $P_{2,n-6} \rightarrow P_{1,n-6}$ that there exists a class $\theta \in H^{2n-6}(P_{2,n-6})$ such that

$$i^*_{2,n-6}(\theta) = Sq^2Sq^4\iota_{2n-12} + (Sq^4 + Sq^3Sq^1)\iota^1_{2n-10} .$$

Observe that $j_2^*\zeta^*(\theta)$ is of the form $\tilde{\theta}\otimes 1$ for some class $\tilde{\theta} \in H^{2n-6}(G_2)$ which is a representative for a secondary cohomology operation Γ_7 associated with the relation

(4.2.11) $$(Sq^2Sq^4)Sq^2 + (Sq^4+Sq^3Sq^1)Sq^4 = 0 .$$

Let $\eta : G_2\times K_{2n-11}\times K^2_{2n-10}\times K_{2n-9}\times K_{2n-7} \longrightarrow G_2$ be the projection .

Then $$\xi_3^*\, j_2^*\,\zeta^*(\theta) = \xi_3^*\,\eta^*(\tilde{\theta}) \in \Gamma_7(b_{n-7}*b_{n-7}) .$$

It is well known that there is a homotopy equivalence

$$\nu : Y_{n-7}*Y_{n-7} \longrightarrow \Sigma(Y_{n-7}\wedge Y_{n-7}) .$$

Therefore $\Gamma_7(b_{n-7}*b_{n-7}) = \nu^*\Gamma_7(b)$ where b is the unique class belonging to $H^{2n-6}(\Sigma(Y_{n-7}\wedge Y_{n-7}))$ such that $s(b) = b_{n-7}\otimes b_{n-7}$. Now a Cartan formula for Γ_7 (see Milgram [22]) gives that

(4.2.12) $$\Gamma_7(b_{n-7}\otimes b_{n-7}) = \Gamma_7(b_{n-7})\otimes b_{n-7} + b_{n-7}\otimes\Gamma_7(b_{n-7})$$

modulo indeterminacies of both sides. We shall need this fact later and it is stated here to anticipate what follows.

It is not hard to deduce from the Serre exact sequence for the fibration $D_{n-7} \longrightarrow K_{n-7}$ that a representative for the operation Γ_7 is given uniquely by

$$(g_4+ Sq^3g_3+ Sq^3Sq^1g_2) \in H^n(D_{n-7}).$$

Thus a representative for the operation Γ_7 on the fundamental class b_{n-7} is given by

$$(h_4 + Sq^3h_3) \in H^n(Y_{n-7})$$ modulo indeterminacy of Γ_7.

Now the total indeterminacy for (4.2.12) is the following direct sum of groups:

$$< b_{n-7}\otimes Sq^2Sq^4h_1>\oplus< Sq^2Sq^4h_1\otimes b_{n-7}>\oplus< b_{n-7}\otimes \chi Sq^4Sq^2h_1>$$
$$\oplus<\chi Sq^4Sq^2h_1\otimes b_{n-7}>$$

where we write χSq^4 for $(Sq^4+Sq^3Sq^1)$ so that we can consider χ as the antiautomorphism of the Steenrod algebra $\mathfrak{A}(2)$. of Milnor . Because $\xi_2^* \zeta^*(\theta)$ lies in the image of i^* by (4.2.12)

$$\begin{aligned}\xi_2^*\zeta^*(\theta) = {} & (h_4+Sq^3h_3)*b_{n-7} + b_{n-7}*(h_4+Sq^3h_3) \\ & + \alpha_1(b_{n-7}*Sq^2Sq^4h_1+Sq^2Sq^4h_1*b_{n-7}) \\ & + \alpha_2(b_{n-7}*\chi Sq^4Sq^2h_1 + \chi Sq^4Sq^2h_1*b_{n-7})\end{aligned}$$

for some $\alpha_1, \alpha_2 \in Z_2$.

Since
$$Sq^2Sq^4(b_{n-7}\cup h_1) = b_{n-7}\cup Sq^2Sq^4h_1 \quad \text{and}$$
$$\chi Sq^4(b_{n-7}\cup Sq^2h_1) = b_{n-7}\cup \chi Sq^4Sq^2h_1 \quad ,$$
$$i^*(Sq^2Sq^4(s^{-1}(b_{n-7}\cup h_1)))=b_{n-7}*Sq^2Sq^4h_1 + Sq^2Sq^4h_1*b_{n-7}$$
$$i^*(\chi Sq^4(s^{-1}(b_{n-7}\cup Sq^2h_1))) = b_{n-7}*\chi Sq^4Sq^2h_1 + \chi Sq^4Sq^2h_1*b_{n-7} \; .$$
by 3.1.2(1).

This implies that we can alter the lifting ξ_1 of s_1 to $P_{2,n-6}$ such that

$$\begin{aligned} i^*\xi_1^*(\theta) &= b_{n-7}*(h_4+Sq^3h_3) + (Sq^3h_3+h_4)*b_{n-7} \\ &= i^*(\gamma) \qquad (\text{by } 3.1.2(1).)\end{aligned}$$

where $\gamma \in H^{2n-6}(\Sigma Y_{n-7})$ is the unique class such that $s(\gamma) = b_{n-7}\cup(h_4+Sq^3h_3)$. Thus by the exactness of the Serre sequence for the fibration $h: \Sigma Y_{n-7} \longrightarrow Y_{n-6}$,

$$\xi_1^*(\theta) = b_{n-7}\cup(h_4+Sq^3h_3) + h^*(x)$$

for some cohomology class $x \in H^{2n-6}(Y_{n-6})$. Now x can be written as the sum of a decomposable class and an indecomposable class $y \in H^{2n-6}(Y_{n-6})$. Define $\hat{\theta} = \theta - p_{2,n-6}^*p_{1,n-6}^*(y)$. Then $\xi_1^*(\hat{\theta}) = b_{n-7}\cup(h_4+Sq^3h_3)$. Therefore $\hat{\theta} \in H^{2n-6}(P_{2,n-6})$ is a natural candidate for the tertiary operation satisfying the conclusion of the theorem. It is now routine to check that a stable class arises this way . This completes the proof of Theorem 4.2.9 .

4.2.13. The indeterminacy of Ⓗ.

The indeterminacy of Ⓗ by Lemma 4.1.3 is given by the secondary operation on all cohomology vector

$$(a_1, a_2, a_3, a_4, a_5) \in H^{k+n-7}(X) \oplus H^{k+n-6}(X) \oplus H^{k+n-4}(X) \oplus \oplus H^k(X) \oplus H^{k+1}(X)$$

where X is the space operated on and k is the dimension of the domain,

assoociated with the following portion of the chain complex $\mathcal{C}_n$:

(4.2.14) $$C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \quad .$$

A particular tertiary cohomology operation is associated with its geometrical realization. Take a geometrical realization as given by Lemma 4.1.3 :

(4.2.15)
$$\begin{tikzcd}
\Omega M_{2,k} \arrow[r, equal] \arrow[d, "i_{3,k}"'] & \Omega M_{2,k} \arrow[d, "i_{2,k}"] & \\
G_{2,k} \arrow[r, "\varsigma_k"] \arrow[d, "p_{3,k}"'] \arrow[dr, phantom, "1"] & P_{2,k} \arrow[r, dash] \arrow[d, "p_{2,k}"] & K_{k+n} \\
\Omega M_{1,k} \arrow[r, "i_{1,k}"'] & P_{1,k} \arrow[r] \arrow[d, "p_{1,k}"] & M_{2,k} \\
& K_k \arrow[r] & M_{1,k}
\end{tikzcd}$$

where $M_{1,k} = K_{k+1} \times K_{k+2} \times K_{k+n-6} \times K_{k+n-5} \times K_{k+n-3}$,

$M_{2,k} = K_{k+n-5} \times K_{k+n-4} \times K^1_{k+n-3} \times K^2_{k+n-3} \times K_{k+n-2} \times K_{k+n} \times K_{k+3}$,

$p_{1,k}: P_{1,k} \to K_k$ is the principal fibration classified by

$$(Sq^1 \iota_k, Sq^2 \iota_k, Sq^{n-6} \iota_k, Sq^{n-5} \iota_k, Sq^{n-3} \iota_k) ,$$

$p_{2,k}: P_{2,k} \longrightarrow P_{1,k}$ is the principal fibration classified by

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \psi_1)$$

where $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \psi_1 \in H^*(P_{1,k})$ are respectively the representatives for the operations Φ_1, Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 and Ψ_1 ,

and square 1 is a fibre square,

$i_{1,k}$, $i_{2,k}$ and $i_{3,k}$ are the respective fibre inclusions.

Then the indeterminacy $\mathrm{Indet}^{k+n}(X, \Theta)$ for Θ with $\theta \in H^{k+n}(P_{2,k})$ as its representative is the following union:

$$\bigcup \{ f^*(\zeta^*(\theta)) \mid \text{for all lifting } f: X \to G_{2,k} \text{ of } (a_1, a_2, a_3, a_4, a_5) \text{ satisfying}$$

$$\begin{aligned}
& Sq^2 a_3 + Sq^{n-5} a_1 = 0 , \\
& Sq^2 a_4 + Sq^{n-4} a_1 = 0 , \\
& Sq^4 a_3 + Sq^{n-3} a_1 + Sq^{n-4} a_2 = 0 , \\
& Sq^2 Sq^1 a_4 + Sq^1 a_5 + Sq^{n-3} a_1 = 0 , \\
& Sq^4 a_4 + Sq^{n-3} a_2 = 0 , \\
& Sq^4 a_5 + Sq^{n-1} a_2 + Sq^n a_1 = 0 \text{ and} \\
& Sq^3 a_1 + Sq^2 a_2 = 0 \quad \}
\end{aligned} \tag{4.2.16}$$

where the union is taken over all such cohomology vectors $\in H^k(X) \oplus H^{k+1}(X) \oplus H^{k+n-7}(X) \oplus H^{k+n-6}(X) \oplus H^{k+n-4}$.

§ 4.3. Cohomology of $BSpin_k[8]$

We shall in this section digress into the theory of the cohomology structure of $BSpin_k[8]$, the 7-connective covering over $BSPin_k$. Our main contention is that everything we have done so far for the fibration $BSpin_{n-7} \longrightarrow BSpin_n$ when $n > 15$ goes as well for the fibration $BSpin_{n-7}[8] \longrightarrow BSpin_n[8]$ when n is greater than 23 .

4.3.1. Following A. Borel [8], we shall investigate the modulo 2 cohomology structure of $BSpin_k[8]$ using the fibration

$$K_3^* \longrightarrow BSpin_k[8] \longrightarrow BSpin_k$$

which kills the first non-zero homotopy group of $BSpin_k$.

It is well known that the mod 2 cohomology ring of K_3^* is a polynomial algebra generated by the following

$$\{ Sq^I \iota_3 \mid I = (i_1, i_2, i_3, \ldots, i_r) \text{ is an admissible sequence with excess} < 3 \text{ and the last index not equal to } 1 \text{ ; the excess of I } e(I) = 2i_1 - \sum_{j=1}^{j=r} i_j \text{ . } \} .$$

We set up the following notation:

$X_r = Sq^{2^r} Sq^{2^{r-1}} \ldots . Sq^{2^1} \iota_3$;

$z = \iota_3$ the fundamental class of K_3^* ;

$v_i = w_i$ if $i \leqslant k$ and i not of the form $2^p + 2^q + 1$;

$$v_{2^r+1} = Sq^{2^{r-1}}Sq^{2^{r-2}}\dots Sq^2Sq^1w_2 \text{ if } 2^r+1 \leqslant k ;$$

$$v_{2^{t+j}+2^j+1} = Sq^{(2^t+1)2^{j-1}}\dots Sq^{2^t+1}Sq^{2^{t-1}}\dots Sq^4Sq^2w_4$$

$$\text{if } j > 0 \text{ and } 2^{t+j}+2^j+1 \leqslant n ;$$

$$v_{2^t+2} = Sq^{2^{t-1}}Sq^{2^{t-2}}\dots Sq^4Sq^2w_4 .$$

These are formal symbols which suggest that we are working in the cohomology structure of BSO_k and do the modifications needed to give some information about $H^*(BSpin_k)$.

Observe that

$\{ z, z^2, z^4, \dots, z^{2^r}, \dots\}$ is a simple systems of generators in the sense of Borel[8], and

$\{X_r, X_r^2, \dots, X_r^{2^j}, \dots\}$ is also a simple systemsof generators for each $r > 0$. These together form a simple system of generators for $H^*(K_3^*)$.

For each k denote s(k) to be the integer such that

$$2^{s(k)-1} < k \leqslant 2^{s(k)}$$

and $\rho(k)$ to be the integer such that

$$2^{s(k)-1} + 2^{\rho(k)-1} < k \leqslant 2^{s(k)-1} + 2^{\rho(k)}.$$

For $t > j$ let $L(t,j)$ be the ideal generated by

$$\{ v_{2^s+2^l+1} \mid s > l \geqslant 0,\ s < t \text{ or } s=t ,\ l \leqslant j \} .$$

For $t = j$ as convention we set $L(t,t) = L(t+1,0)$.

4.3.2. The spectral sequence for the fibration

(4.3.3) $K_3^* \longrightarrow BSpin_k[8] \longrightarrow BSpin_k$.

Denote by $\{ E_r, d_r\}$ the spectral sequence for the fibration (4.3.3) .

4.3.4. PROPOSITION. There exists the following isomorphisms

$$E_{2^{s(k)-1}+2^{\rho(k)-1}+2} \cong H^*(BSpin_k) \,/\, L(s(k)-1, \rho(k)-1) \otimes$$

$$(4.3.5) \quad \otimes Z_2\left[X_{s(k)-\rho(k)-1}^{2^{\rho(k)}}, X_{s(k)-\rho(k)-2}^{2^{\rho(k)+1}}, \ldots, X_1^{2^{s(k)-2}} ; z^{2^{s(k)-2}} ;\right.$$

$$\left. X_{s(k)-2}^{X_2 1}, \ldots x, X_{s(k)-\ (k)}^{2^{\rho(k)-1}} ; X_{s(k)-3}, X_{s(k)-2}, \ldots\right].$$

In particular $E_{2^{s(k)-1}+2^{\rho(k)}+1} \cong E_{2^{s(k)-1}+2^{\rho(k)-1}+2}$.

Proof. According to Serre we have the following additive isomorphism: $E_2^{*,*} \cong H^*(K_3^*) \otimes H^*(BSpin_k)$.

The first non-zero differential is d_4 and $d_4(z) = v_4$. Because

$$E_4 \cong E_2 \cong H^*(BSpin_k) \otimes H^*(K_3^*)$$

$$\cong H^*(BSpin_k) \otimes Z_2[z] \otimes Z_2[X_1, X_2, \ldots,] \quad ,$$

therefore $E_5 \cong H^*(BSpin_k)/ L(1,0) \otimes Z_2[X_1, X_2, \ldots .]$

by the Kunneth theorem .

For convenience we denote elements in $E_j^{*,*}$ by its counter image in $E_2^{*,*}$. Thus

$$d_{2^{j+1}+2}(1 \otimes X_j) = v_{2^{j+1}+2} \quad \text{for } j < s(k)-2;$$

$$d_{2^t(2^{j+1}+1)}(1 \otimes X_j^{2^t}) = v_{2^t(2^{j+1}+1)+1}$$

$$\text{if } t+j < s(k)-2 \text{ or } t+j = s(k)-2,\ t < \rho(k) .$$

(Note that dimension of $X_j^{2^t} = 2^t(2^{j+1}+1)$.)

So this implies (usin- the Kunneth theorem succcsively)

$$E_{2^t(2^{j+1}+1)+2} \cong H^*(BSpin_k)/L(t+j+1,t) \otimes$$

$$\otimes Z_2\left[X_j^{2^{t+1}} ; X_{j-1}^{2^{t+2}}, \ldots x X_1^{2^{t+j}} ; z^{2^{t+j-1}} ;\right.$$

$$\left. X_{j+t}^2, \ldots, X_{j+1}^{2^t} \right] \otimes Z_2\left[X_{j+t+1}, X_{j+t+2}, \ldots, \ldots\right]$$

for $2^t(2^{j+1}+1)+2 \leq n$. Thus the proposition follows. The last assertion follows since each of the generator of $H^*(K_3^*)$ in the expression (4.3.5) has dimension $\geqslant 2^{s(k)} \geqslant n$. This completes the proof of Proposition 4.3.4 .

4.3.6. For the moment we regard $BSpin_k$ as if it were BSO_k . Our goal is the following:

PROPOSITION. $H^*(BSpin_k[8])$ is a polynomial algebra iff n is less than 20.

Proof. First we set up some notation. For each integer $j > 1$ let $I(j)$ be the ideal generated by $\{ v_2, v_3, \ldots v_j \}$.

We note the following easily proven facts:

(4.3.7) If $x \in I(j)$ and dimension of x is a then

$$Sq^{a-1}x,\ Sq^{a-2}x \in I(2j-1);$$

(4.3.8) if $x \in L(t,j)$ for some $t > j$ and the dimension of x is a then

$$Sq^{a-1}x,\ Sq^{a-2}x \in L(t+1,j+1) .$$

The following formulae of Wu is used in the proof of Proposition 4.3.6 and frequently in the remainder of this section:

(4.3.9) $$Sq^{j-1}w_j = w_{j-1}\cdot w_j + w_{j-2}\cdot w_{j+1} + \cdots + w_2\cdot w_{2j-3} + w_{2j-1} ;$$

(4.3.10) $$Sq^{2n-2}w_{2n} = w_{2n-2}\cdot w_{2n} + w_{2n-4}\cdot w_{2n+2} + \cdots + w_2\cdot w_{4n-4} + w_{4n-2}.$$

We can now begin the proof properly.

Now

$$v_{2^{s(k)-2}+2} = Sq^{2^{s(k)-3}} Sq^{2^{s(k)-4}} \ldots\ldots Sq^2 w_4$$

$$= Sq^{2^{s(k)-3}} v_{2^{s(k)-3}+2}$$

$$= w_{2^{s(k)-2}+2} + Q + v_{2^{s(k)-3}} \cdot R$$

where $Q \in I(2^{s(k)-3}-1)$ and R is of degree $2^{s(k)-3}+2$. This is easily seen from (4.3.9), (4.3.10) and (4.3.7).

Therefore,

$$Sq^{2^{s(k)-2}} v_{2^{s(k)-2}+2} = w_{2^{s(k)-1}+2} +$$

$$+ \sum_{0<t<2^{s(k)-3}} w_{2^{s(k)-2}-2t} \cdot w_{2^{s(k)-2}+2(t+1)}$$

$$+ Sq^{2^{s(k)-2}} Q + Sq^{2^{s(k)-3}-2} v_{2^{s(k)-3}} \cdot R^2 +$$

$$+ Sq^{2^{s(k)-3}-1} v_{2^{s(k)-3}} \cdot Sq^{2^{s(k)-3}+1} R$$

$$+ (v_{2^{s(k)-3}})^2 \cdot Sq^{2^{s(k)-3}} R$$

$$\equiv w_{2^{s(k)-1}+2} + w_{2^{s(k)-2}} \cdot w_{2^{s(k)-2}+2} +$$

$$+ w_{2^{s(k)-2}-2} \cdot w_{2^{s(k)-2}+4}.$$

$$\text{modulo } I(2^{s(k)-2}-3) \quad \ldots. \quad (4.3.11).$$

Also,

$$v_{2^{s(k)-2}+2^{\rho(k)-1}+1} = w_{2^{s(k)-2}+2^{\rho(k)-1}+1} + Q +$$

$$v_{2^{s(k)-3}+2^{\rho(k)-2}} \cdot R$$

where $Q \in I(2^{s(k)-3}+2^{\rho(k)-1}-1)$, R is of degree $2^{s(k)-3}+2^{\rho(k)-2}+1$.

So,

$$Sq^{2^{s(k)-2}+2^{\varrho(k)-1}}\, v_{2^{s(k)-2}+2^{\varrho(k)-1}+1}$$

$$= Sq^{2^{s(k)-2}+2^{\varrho(k)-1}}\left(w_{2^{s(k)-2}+2^{\varrho(k)-1}+1} + Q\right)$$

$$+ R^2 . Sq^{2^{s(k)-3}+2^{\varrho(k)-2}-1}\, v_{2^{s(k)-3}+2^{\varrho(k)-2}}$$

$$+ \left(v_{2^{s(k)-3}+2^{\varrho(k)-2}}\right)^2 . Sq^{2^{s(k)-3}+2^{\varrho(k)-2}} R$$

$$\equiv w_{2^{s(k)-1}+2^{\varrho(k)}+1} + w_{2^{s(k)-2}+2^{\varrho(k)-1}} . w_{2^{s(k)-2}+2^{\varrho(k)-1}+1}$$

$$+ w_{2^{s(k)-2}+2^{\varrho(k)-1}-1} . w_{2^{s(k)-2}+2^{\varrho(k)-1}+2}$$

$$\text{modulo } I(2^{s(k)-2}+2^{\,(k)-1}-2) \quad \ldots\ldots\ldots \quad (4.3.12) .$$

Thus if $\varrho(k) \geqslant 2$ and $s(k) \geqslant 5$ an application of (4.3.11) to $Sq^{2^{s(k)-1}}\, v_{2^{s(k)-1}+2}$ shows that there is a relation in $H^*(BSpin_k[8])$. Hence the necessity part of Proposition 4.3.6 follows. To show that there is no relation in $H^*(BSpin_k[8])$ when $k < 20$ it is enough to note the following congruences:

$$v_4 = w_4;\ v_6 = Sq^2 w_4 = w_6;\ v_7 = Sq^3 w_4 = w_7\ ;$$

$$v_{13} = Sq^6 Sq^3 w_4 \equiv w_{13} \mod L(2,0);$$

$$v_{10} = Sq^4 Sq^2 w_4 \equiv w_{10} \mod L(1,0);$$

$$v_{11} = Sq^5 Sq^2 w_4 \equiv w_{11} \mod L(2,1);$$

$$v_{18} = Sq^8 Sq^4 Sq^2 w_4 \equiv w_{18} \mod L(3,0);$$

$$v_{19} = Sq^9 Sq^4 Sq^2 w_4 \equiv w_{19} \mod L(3,1);$$

$$v_{21} = Sq^{10} Sq^5 Sq^2 w_4 \equiv w_{21} + w_8 . w_{13} \mod L(3,1) \equiv w_{21} \mod L(3,2);$$

$$v_{25} = Sq^{12} Sq^6 Sq^3 w_4 \equiv w_{25} + (w_8 . w_{17}) \mod L(3,2);$$

$$v_{34} = Sq^{16} Sq^8 Sq^4 Sq^2\, w_4 \equiv w_{34} + w_{14} . w_{20} + w_{12} . w_{22} + w_8 . w_{26} \mod L(4,1)\ ;$$

$$v_{37} = Sq^{18}Sq^{9}Sq^{4}Sq^{2}w_4 \equiv w_{37}+w_{17}\cdot w_{20}+w_{16}\cdot w_{21}+w_{15}\cdot w_{22}+ +w_{14}\cdot w_{23}+w_{12}\cdot w_{25}+w_8\cdot w_{29} \mod L(4,0).$$

When $k \leq 19$ these are all zero modulo the respective L' groups. This shows that any relation that arises in $H^*(BSpin_k)$ for k greater than nine and less than twenty has its relation ideal lying in the L' group. This shows that $H^*(BSpin_k[8])$ is a polynomial algebra when $k \leq 19$. This completes the proof of Proposition 4.3.6.

4.3.13. COROLLARY.

$$H^*(BSpin_{15}[8]) \cong Z_2[w_8, w_{12}, w_{14}, w_{15}, \gamma_{128}, \theta_{20}, \theta_{24}, \theta_{18}, : \psi_{2^{j+1}+1}, j \geq 3].$$

Proof. Following A. Borel we have the following additive isomorphism (after some very tedious computations) :

$$H^*(BSpin_{15}) \cong Z_2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, \gamma_{128}]$$

modulo the ideal generated by †

$$\{ v_{17} = (w_7\cdot w_{10}+w_6\cdot w_{11}+w_4\cdot w_{13}),\ Sq^{16}v_{17},\ Sq^{32}Sq^{16}v_{17} \}.$$

Now $d_{25}(1\otimes z^8) = Sq^{12}v_{13} \equiv w_{25} \mod L(3,2) \equiv 0 \mod L(3,2)$,

$d_{19}(1\otimes X_2^2) \equiv 0 \mod L(3,1)$,

$d_{18}(1\otimes X_3) = Sq^8 v_{10} \equiv 0 \mod L(3,2)$ and

$d_{21}(1\otimes X_1^4) = Sq^{10}v_{11} \equiv 0 \mod L(3,2)$.

So $Sq^{16}Sq^8 v_{10} \in L(4,1)$. Inductively we have

$d_{2^{j+1}+2}(1\otimes X_j) \equiv 0 \mod L(3,2)$ for $j \geq 3$.

This gives then a family $\{ \psi_{2^{j+1}+1} \}_{j \geq 3}$ of generators which survive to the E_∞ term. θ_{20}, θ_{18}, θ_{24} correspond to the vanishing of X_1^4, X_2^2 and z^8 under the differentials. This completes the proof of the Corollary.

† D.G. Quillen had computed the groups $H^*(BSpin_k)$ for all k in ' The Mod 2 Cohomology Rings of Extra-special 2-groups and the Spinor Groups, __ Math. Ann. 194(1971),pp 197-212'.

4.3.14. The spectral sequence for the fibration

$$K(Z_2,1) \longrightarrow BSpin_k \longrightarrow BSO_k$$

For each integer j greater than zero set $L(j)$ to be the ideal generated by $\{ v_{2^t+1} \}_{0 \leq t < j}$. Denote by x the fundamental class of $K(Z_2,1)$. Recall the following:

PROPOSITION (A. BOREL [8]).

Let $\{ D_j , d_j \}$ be the Serre spectral sequence for the fibration $K(Z_2,1) \longrightarrow BSpin_k \longrightarrow BSO_k$. Then additively, for each integer j with $2 \leq j \leq 2^{s(k)}$,

$D_j \simeq D_{j+1}$ if j is not of the form 2^t+1 and

$D_{2^t+2} \simeq H^*(BSO_k)/L(t+1) \otimes Z_2[x^{2^{t+1}}]$ when $0 < t < s(k)$.

From this proposition it is seen that the map

$$\pi : BSpin_{n-k} \longrightarrow BSpin_n$$

does not necessarily induce an epimorphism in cohomology. However for $n \equiv 7\ (8) > 15$ and $k = 7$, $\pi^*: H^*(BSpin_n) \longrightarrow H^*(BSpin_{n-k})$ is an epimorphism in dimensions $< 2^{s(n)}$. This follows from the following considerations:

Let $\{ {}^jD_r , d_r \}$ be the spectral sequence for $BSpin_j$. Since $n \equiv 7(8) > 15, s(n-7) = s(n)$. So by Proposition 4.3.14 and the fact that $\pi^*: H^*(BSO_n) \longrightarrow H^*(BSO_{n-7})$ is an epimorphism in all dimensions that the induced homomorphism

$${}^nD_{2^{s(n)-1}+2} \longrightarrow {}^{n-7}D_{2^{s(n)-1}+2}$$

is always an epimorphisms in all dimensions if n is not of the form $7 + 2^{s(n)-1}$.

Suppose now that $n = 7+2^{s(n)-1}$, $s(n) > 4$. Proposition 4.3.14 then says that the first relation which consists of decomposables for $H^*(BSpin_{n-7})$ occurs in dimension $2^{s(n)-1}+1$ and that for $H^*(BSpin_n)$ occurs in dimension $2^{s(n)}+1$. Since $s(n) > 4$ the first non characteristic class of $H^*(BSpin_{n-7})$ occurs in dimension $\geqslant 2^{s(n)}$ whereas the first non-characteristic class of $H^*(BSpin_n)$ occurs in dimension $\geqslant 2^{s(n)+1}$. This implies that the homomorphism

$$\pi^*:H^*(BSpin_n) \longrightarrow H^*(BSpin_{n-7})$$

is an epimorphism in dimensions $< 2^{s(n)}$. We summarize the above deduction as follows:

4.3.15. LEMMA. Suppose $n \equiv 7(8) > 15$. Then

$$\pi^*:H^*(BSpin_n) \longrightarrow H^*(BSpin_{n-7})$$

is an epimorphism in dimensions $< 2^{s(n)}$. Moreover in dimensions $< 2^{s(n)}$ the $\mathcal{A}(2)$-module structures of $H^*(BSpin_n)$ and $H^*(BSpin_{n-7})$ are entirely given by the Wu formula.

4.3.16. ADDENDUM. Additively,

$$H^*(BSpin_{15}) \cong Z_2[w_4,w_6,w_7,w_8,w_{10},w_{11},w_{12},w_{13},w_{14},w_{15},\gamma_{128}]$$

modulo the ideal generated by

$$\{ v_{17}=(w_7\cdot w_{10}+w_6\cdot w_{11}+w_4\cdot w_{13}),\ Sq^{16}v_{17},\ Sq^{32}Sq^{16}v_{17}\},$$

$H^*(BSpin_8) \cong Z_2[w_4,w_6,w_7,w_8,\gamma_8]$. Therefore

$$\pi^*:H^*(BSpin_{15}) \longrightarrow H^*(BSpin_8)$$

is an epimorphism in dimensions < 12 and not equal to 8.

Addendum 4.3.16 is an easy application of Proposition 4.3.14. Therefore by Proposition 4.3.4 thereis an additive isomorphism

$$H^*(BSpin_8[8]) \cong Z_2[w_8, \eta_8, \theta_{10}, \theta_{12}; \{\psi_{2^{j+1}+1}\}_{j\geq 2}].$$

Thus $\pi^*: H^*(BSpin_{15}[8]) \longrightarrow H^*(BSpin_8[8])$ is an epimorphism in dimensions < 16 and not equal to any of $\{8, 9, 10, 12\}$. by comparing with the isomorphism of 4.3.13.

4.3.17. PROPOSITION. Suppose $n \equiv 7\ (8) > 15$ and n is not of the form $(7+2^{s(n)-1})$. Then

$\pi^*: H^*(BSpin_n[8]) \longrightarrow H^*(BSpin_{n-7}[8])$ is an epimorphism in dimensions $< 2^{s(n)}$.

Proof. Let $\{ {}^jE_r, d_r \}$ denote the spectral sequence for $H^*(BSpin_j[8])$. Since $n \equiv 7\ (8) > 15$ and n is not of the form $7+2^{s(n)-1}$, $\rho(n) \neq 3$, $s(n) = s(n-7)$, $\rho(n) = \rho(n-7)$. It follows from 4.3.5 and Proposition 4.3.4 that the induced homomorphism, ${}^nE_{2^{s(n)-1}+2^{\rho(n)-1}+2} \longrightarrow {}^{n-7}E_{2^{s(n)-1}+2^{\rho(n)-1}+2}$ is an epimorphism in dimensions $< 2^{s(n)}$. The conclusion then follows from this.

4.3.18. PROPOSITION. Suppose $n = 7 + 2^{s(n)-1}$, $s(n) > 5$. Then $\pi^*: H^*(BSpin_n[8]) \longrightarrow H^*(BSpin_{n-7}[8])$ is an epimorphism in dimensions $< 2^{s(n)}$.

Proof. Since $n = 7 + 2^{s(n)-1}$ and $s(n) > 5$, $\rho(n) = 3$, and $s(n) = s(n-7)$. Observe that $\rho(n-7) = s(n)-2$. By Proposition 4.3.4 the first new relation which consists of decomposables for $H^*(BSpin_{n-7}[8])$ occurs in dimension $2^{s(n)-1}+2$; therefore the first non-characteristic class not lying in $H^*(BSpin_{n-7})$, is of degree $> 2^{s(n)}+2$. Similarly the first new relation which consists entirely of decomposables for $H^*(BSpin_n[8])$ occurs in dimension $2^{s(n)-1}+2^3+1$ by (4.3.9), (4.3.10) and (4.3.12). Therefore the first new non-characteristic class lies in dimension $\geqslant 2^{s(n)}+2^4$. Thus by Lemma 4.3.15 and Proposition 4.3.4, $\pi^*: H^*(BSpin_n[8]) \longrightarrow H^*(BSpin_{n-7}[8])$ is an epimorphism in dimensions $< 2^{s(n)}$. This completes the proof of Proposition 4.3.18.

4.3.19. ADDENDUM. Additively there are isomorphisms:

$$H^*(BSpin_{16}[8]) \cong Z_2[w_8, w_{12}, w_{14}, w_{15}, w_{16}, \eta_{2^7} ; \theta_{18}, \theta_{20}, \theta_{24}; \{ \psi_{2^{j+1}+1} \mid j \geqslant 3 \}]$$

where θ_{18}, θ_{20}, θ_{24}, $\{\psi_{2^{j+1}+1}\}_{j \geqslant 3}$ correspond respectively to the vanishing of X_2^2, X_1^4, z^8 and X_j $j \geqslant 3$ under the differentials;

$$H^*(BSpin_{23}[8]) \cong Z_2[w_8, w_{12}, w_{14}, w_{15}, w_{16}, w_{20}, w_{22}, w_{23}, \eta_{2^{11}} ; \tilde{\theta}_{24}, \tilde{\theta}_{40}, \tilde{\theta}_{72}, \tilde{\theta}_{129}, \tilde{\theta}_{130}, \tilde{\theta}_{132}, \tilde{\theta}_{136} ; \{ \tilde{\psi}_{2^{j+1}+1} \mid j \geqslant 7\}]$$

modulo the ideal generated by

$$\{(w_{14} \cdot w_{20} + w_{12} \cdot w_{22}), (w_{15} \cdot w_{20} + w_{12} \cdot w_{23}), (w_{15} \cdot w_{22} + w_{14} \cdot w_{23}),$$
$$(w_{15}^2 \cdot w_{16} \cdot w_{23} + w_{23}^2 \cdot w_8 \cdot w_{15} + w_{23}^3),$$
$$(w_{14}^2 \cdot w_{16} \cdot w_{23} + w_{22}^2 \cdot w_{23} + w_{22}^2 \cdot w_8 \cdot w_{15}),$$
$$(w_{14}^2 \cdot w_{16} \cdot w_{22} + w_{22}^2 \cdot w_8 \cdot w_{14} + w_8 \cdot w_{12} \cdot w_{23}^2 + w_{15}^2 \cdot w_{16} \cdot w_{20} + w_{23}^2 \cdot w_{20} + w_{22}^3)\}$$

where $\tilde{\theta}_{24}$, $\tilde{\theta}_{40}$, $\tilde{\theta}_{72}$, $\tilde{\theta}_{129}$, $\tilde{\theta}_{130}$, $\tilde{\theta}_{132}$, $\tilde{\theta}_{136}$, $\tilde{\psi}_{2^{j+1}+1}$ $j \geqslant 7$ correspond to the vanishing of z^8, X_1^8, X_2^8, X_6, X_5^2, X_4^4, X_3^8, X_j $j \geqslant 7$ under the differentials.

Proof. This follows from an easy argument using 4.3.14, and Proposition 4.3.4. Quillen's computation of $H^*(BSpin_k)$ gives the dimensions of η_{2^7}, $\eta_{2^{11}}$. The details are left to the reader.

§4.4. The Single Obstruction.

We apply the result of Chapter two to the calculation of $\Theta^*((T\pi)^*(U_{BSpin_n[8]}))$ where Θ^* is a tertiary cohomology operation valid only on integral classes and is associated with the chain complex $\mathcal{C}_n$ and whose domain is the kernel of $(\Phi_1^*, \Phi_2^*, \Phi_3^*, \Phi_4^*, \Phi_5^*, \Phi_6^*, \psi_1^*)$. ($\Theta^*$ is associated with the relation

$$(4.1.8)^* \qquad Sq^2Sq^4\Phi_1^* + \chi(Sq^4)\Phi_3^* + Sq^{n-2}\psi_1^* = 0\ . \quad)$$

The following is immediate from Theorem 4.2.9:

4.4.1. THEOREM. For $n \equiv 7(8) \geq 15$

$$U_{BSpin_{n-7}[8]} \cup (\Phi_{2,2}(U_{BSpin_{n-7}[8]}) + Sq^3\Phi_{2,0}(U_{BSpin_{n-7}[8]})) \in s^7\Theta(s^{-7}(U_{BSpin_{n-7}[8]}).$$

Hence $0 \in \Theta(U_{BSpin_{n-7}[8]})$.

Thus

4.4.2. COROLLARY. $0 \in \Theta^*((T\pi)^*(U_{BSpin_n[8]}))$.

Proof. This follows from Theorem 4.4.1 and Lemma 3.1.7.

We may now state the main theorem of this section.

4.4.3. THEOREM. Let M be a closed and connected manifold of dimension $n \equiv 7(8) \geq 15$. Suppose $4 \leq r \leq 7$. Suppose M is $(r-2)$-connected mod 2 and ξ is an n-plane bundle over M satisfying either $\delta w_{n-r}(\xi) = 0$ when n-r is even or $w_{n-r+1}(\xi) = 0$ when n-r is odd. Then the following statements hold:

(1) If $4 \leq r < 6$ and $w_4(\xi) \neq w_4(M)$ then ξ has r independent cross sections.

(2) Suppose $4 \leq r \leq 6$ and $w_4(\xi) = w_4(M)$, then ξ has r independent cross sections if and only if $0 \in \Phi_6^*(U_\xi)$ where U_ξ is the Thom class of ξ.

(3) If $r = 7$ suppose the first Pontrjagin class of ξ $P_1(\xi)$ is zero . Then ξ has 7 independent cross sections if and only if $0 \in \Theta^*(U_\xi)$.

Theorem 4.4.3 part (1) and (2) is essentially a special case of Theorem 6.7 of Thomas[32] . We shall prove only part (3).

4.4.4. Suppose $n > 23$. Let the notation be given by 1.2.19 where $BSpin_{n-7}$, $BSpin_n$ are replaced respectively by $BSpin_{n-7}[8]$ and $BSpin_n[8]$. First observe that

$$\mathrm{Indet}((\Phi_1^*, \Phi_2^*, \Phi_3^*, \Phi_4^*, \Phi_5^*, \Psi_1^*), \ TE^1\ ) = \Psi_{E^1}(\mathrm{Indet}(k^2\ )) \qquad \text{...........} \ (4.4.5) .$$

Let Θ be the class of k_n^3 with respect to $\mathrm{Ker}(p_2^*) \cap \mathrm{Im}(q_2^*)$ for a fixed lifting p_2 of $p_1 : BSpin_{n-7}[8] \to E^1$ to E^2. By Theorem3.5.1, (4.4.5) and the fact that $\pi^* : H^*(BSpin_n\ 8\ ) \to H^*(BSpin_{n-7}[8])$ is an epimorphism in dimensions $\leq n$ (see Proposition 4.3.17), k^2 is admissible for k_n^3 with respect to Θ^* and the universal n-plane bundle over BSO_n . Thus an application of Admissible class Theorem 2.2.14 gives the following:

4.4.6. THEOREM. Suppose $n \equiv 7(8) > 23$. There exists a class $e \in \mathcal{O}$ such that

$$\Psi_{E^2}(e) \in \textcircled{H}^*((T(q_1 \circ q_2))^*(U_{BSpin_n[8]})\ .$$

Therefore by Proposition 2.2.10 and the fact that $\mathrm{Ker}(p_2^*) \cap \mathrm{Im}(q_2^*) \cap H^n(E^2) = \{0\}$,

$$\Psi_{E^2}(k_n^3) \in \textcircled{H}^*((T(q_1 \circ q_2))^*(U_{BSpin_n[8]}).$$

Because $BSpin_n[8]$ is 7-connected we see that the following is easily derived at:

4.4.7. PROPOSITION. $\mathrm{Indet}(k_n^3) = \Psi_{E^2}^{-1}(\mathrm{Indet}(\textcircled{H}^*, TE^2)$.

Hence the following:

4.4.8. COROLLARY. For $n > 23$, $\{ U_{E^2} \cup k_n^3 \} = \textcircled{H}^*(U_{E^2})$ where the indeterminacy is of the left hand side.

4.4.9. The case $n = 23$. Let the tower over $BSpin_{23}[8]$ be given by the 'pullback' of the Postnikov tower (1.2.19) for the fibration $\pi : BSpin_{16} \longrightarrow BSpin_{23}$:

(4.4.10)
$$\begin{tikzcd}
& \tilde{E}^2 \arrow[rrr, "f_2"] \arrow[d, "\tilde{q}_2"] \arrow[drrr, phantom, "1"] & & & E^2 \arrow[d, "q_2"] & \\
& \tilde{E}^1 \arrow[rrr, "f_1"] \arrow[dd, "\tilde{q}_1"] & & & E^1 \arrow[r, "k^2"] \arrow[dd, "q_1"] & C_2 \\
BSpin_{n-7}[8] \arrow[uur, "\tilde{p}_2"] \arrow[ur, "\tilde{p}_1"] \arrow[dr, "\pi"] \arrow[rrr, dash] & & & BSpin_{n-7} \arrow[uur, "p_2"] \arrow[ur, "p_1"] \arrow[dr, "\pi"] & & \\
& BSpin_n[8] \arrow[rrr] \arrow[urr, phantom, "2"] & & & BSpin_n \arrow[r, "k^1"] & C_1
\end{tikzcd}$$

where squares 1 and 2 are pullback squares and the rest of the diagram is defined in an obvious way.

By Addendum 4.3.19,

$\mathrm{Indet}(\Phi_i^*, TBSpin_{n-7}[8]) \cong 0$ for $i = 1,2,3,5,6$ and

$\mathrm{Indet}(\Phi_4^*, TBSpin_{n-7}[8], U_{BSpin_{n-7}[8]}) \cong \langle \theta_{20} \rangle_{Z_2} \neq 0$

(because $Sq^2 Sq^1 \psi_3 = \theta_{20}$). The admissible class Theorem 2.2.14 applies to give that

$$\Psi_{E^1}(k_4^2) \in \Phi_4^*((T\pi)^*(U_{BSpin_n})\ .$$

Thus $\quad \Psi_{E^1}(f_1^*(k_4^2)) \in \Phi_4^*((T\pi)^*(U_{BSpin_n[8]})\ .$

Because $Sq^2 Sq^1 \psi_3 \neq 0$ ψ_3 cannot be any part of any element in the domain of $\mathrm{Indet}(\Theta^*)$. Thus

$$\mathrm{Indet}(\Theta^*, (T\pi)^*(UBSpin_{n-7}[8])) = (T\pi)^*(\mathrm{Indet}(\Theta^*, U_{BSpin_n[8]})\,.$$

Therefore $f_1^*(k^2)$ is admissible for $f_2^*(k_n^3)$. Hence Theorem 2.2.14 and Theorem 4.4.1 apply to give

PROPOSITION. Let $\tilde{\sigma}$ be the coset of $f_2^*(k_n^3)$ with respect to $\mathrm{Ker}(\tilde{p}_2^*) \cap \mathrm{Im}((\tilde{q}_1 \circ \tilde{q}_2)^*) \cap H^n(\tilde{E}^2)$. Then there exists a class $e \in \tilde{\sigma}$ such that

$$\Psi_{E^2}(e) \in \Theta^*((T\pi)^*(UBSpin_n[8]))\ .$$

4.4.11. The case $n = 15$.

Let the tower over $BSpin_{15}[8]$ be given by the 'pullback of the Postnikov tower for the fibration $BSO_8 \longrightarrow BSO_{15}$:

(4.4.12)
$$\begin{tikzcd}
 & \overset{\approx}{E}^2 \arrow[r, "\tilde{f}_2"] \arrow[d, "\overset{\approx}{q}_2"] & \tilde{E}^2 \arrow[r, "f_2"] \arrow[d, "\tilde{q}_2"] & E^2 \arrow[d, "q_2"] & \\
BSpin_8[8] \arrow[ur, "\overset{\approx}{p}_2"] \arrow[r, "\overset{\approx}{p}_1"] \arrow[dr, "\pi"'] & \overset{\approx}{E}^1 \arrow[r, "\tilde{f}_1"] \arrow[d, "\overset{\approx}{q}_1"] & \tilde{E}^1 \arrow[r, "f_1"] \arrow[d, "\tilde{q}_1"] & E^1 \arrow[r, "k^2"] \arrow[d, "q_1"] & C_2 \\
 & BSpin_{15}[8] \arrow[r] & BSpin_{15} \arrow[r] & BSO_{15} \arrow[r, "k^1"] & C_1
\end{tikzcd}$$

where $E^2 \longrightarrow E^1 \longrightarrow BSO_{15}$ is a Postnikov tower for the fibration $BSO_8 \longrightarrow BSO_{15}$ through dimension 15 and every square is a pullback square. Note that the k-invariants for the third stage has its n-dimensional component given by the same relation on k^2 with some twisting via w_2.

Now $H^{15}(BSpin_8[8]) \cong 0$ by 4.3.16. Trivially

$$\textcircled{H}^*((T\pi)^*(U_{BSpin_{15}[8]}) = 0$$

modulo zero indeterminacy.

Thus $(f_1 \circ \tilde{f}_1)^*(k^2)$ is admissible for $(f_2 \circ \tilde{f}_2)^*(k_n^3)$. Therefore the Admissible Class Theorem 2.2.14 applies to give

PROPOSITION. Let $\tilde{\tilde{\mathcal{O}}}$ be the coset of $(f_2 \circ \tilde{f}_2)^*(k_n^3)$ with respect to $\mathrm{Ker}(\tilde{\tilde{P}}_2^*) \cap \mathrm{Im}((\tilde{\tilde{q}}_1 \circ \tilde{\tilde{q}}_2)^*) \cap H^{15}(\tilde{\tilde{E}}^2)$. Then there exists a class $e \in \tilde{\tilde{\mathcal{O}}}$ such that

$$\Psi_{\tilde{\tilde{E}}^2}(e) \in \textcircled{H}^*((T\pi)^*(U_{BSpin_{15}[8]}).$$

4.4.13. Proof of Theorem 4.4.3(3).

Since the manifold M by hypothesis is 5-connected and by assumption on the n-plane bundle ξ over M $P_1(\xi) = 0$ ξ admits a $Spin_n[8]$ structure. Part (3) of Theorem 4.4.3 then follows from Corollary 4.4.8, 4.4.9, 4.4.11 and the fact that $\quad \mathrm{Indet}(k_n^3) = \Psi_{E^2}^{-1}(\mathrm{Indet}(\textcircled{H}^*, TE^2)$ in all cases.

This completes the proof of Theorem 4.4.3.

CHAPTER 5. COHOMOLOGY OPERATIONS ON THE THOM COMPLEX AND THE EXISTENCE OF 7-FIELDS

This chapter is mainly computation. The main idea is that the operation Θ^* on the Thom class of the tangent buundle of a manifold satisfying conditions (A) and (B) with $k = 7$ of Chapter 1 is always zero modulo zero indeterminacy. We shall do this by showing that the operation Θ^* on the Thom class U_τ of the tangent bundle of M is the same as another tertiary cohomology operation and that this latter operation on U_τ is always zero. In fact the choice of a representative for Θ^* can be determined by a choice for the new operation $\tilde{\Omega}^*$. $\tilde{\Omega}^*$ is associated with a chain complex obtained by modifying $\mathcal{C}_n$ without altering much about its property as detecting the k-invariants for lifting. Let $g: M\times M \longrightarrow T_\tau$ be the map that collapses the complement of a tubular neighbourhood of the diagonal in $M\times M$ to a point. We shall show that $\tilde{\Omega}^*(g^*(U_\tau))$ is always zero whenever it is defined.

In § 1 we present a small discussion on N^{th} order cohomology operations due to Adams and Maunder [21]. In § 2 we state the main theorem. The proof of the theorem is presented in § 3. The remainder of the chapter is a discussion on the dual cohomology operations associated with $\mathcal{C}_n$ and some observation about the vanishing of the dual cohomology operation on the Thom class of the stable bundle over BSO [8].

§5.1. Axioms for N^{th} order cohomology operations and S-duality.

5.1.1. Definition. Let X be a space.

Let

$$\mathcal{C} : \; C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \longleftarrow \cdots\cdots \; C_r \xleftarrow{d_{r+1}} C_{r+1} \; \cdots \longleftarrow C_{N-1} \xleftarrow{d_N} C_N$$

be a free chain complex of order two of $\mathcal{A}$-modules. where $\mathcal{A}$ is a Steenrod algebra. Then a pyramid of cohomology operations $\Phi^{r,s}$ associated with ($N \geqslant r > s \geqslant 0$) is a family of operations satisfying the following axioms:

<u>Axiom 1</u> (Induction). $\Phi^{r,s}$, $u \geqslant r > s \geqslant v$ is associated with the complex:

$$C_v \longleftarrow \cdots\cdots \quad \longleftarrow C_u \quad .$$

<u>Axiom 2</u> (Domain). $\Phi^{N,0}(\varepsilon)$ is defined for those $\mathcal{A}$-homomorphism $\varepsilon : C_0 \longrightarrow H^*(X)$ such that $\Phi^{i,0}(\varepsilon)$ is defined and is an equivalence class (Axiom 3) containing the zero map for each $i < N$.

<u>Axiom 3</u>. If $\varepsilon : C_0 \longrightarrow H^*(X)$ is an $\mathcal{A}$-map of degree q and $\Phi^{N,0}(\varepsilon)$ is defined, then it is an equivalence class of $\mathcal{A}$-morphisms $\eta : C_N \longrightarrow H^*(X)$ of degree q-N+1 , with respect to the euivalence relation: $\eta \sim \eta'$ if there exists $\mu \in \Phi^{N,1}(\rho)$ such that $\eta' = \eta + \mu$ for some $\rho : C_1 \longrightarrow H^*(X)$ of degree q-1, such that $\Phi^{N,1}(\rho)$ is defined.

<u>Axiom 4</u> (Naturality). If $f:Y \longrightarrow X$ is a map of topological spaces and $\Phi^{N,0}(\mathcal{E})$ is defined for $\mathcal{E}:C_0 \longrightarrow H^*(X)$, then

$$\Phi^{N,0}(f^*\mathcal{E}) \text{ is defined and } f^*\Phi^{N,0}(\mathcal{E}) \subset \Phi^{N,0}(f^*\mathcal{E}) .$$

and

<u>Axiom 5</u> (Peterson-Stein). Let (X,Y) be a pair of spaces and suppose $\mathcal{E}:C_0 \longrightarrow H^*(X)$ is such that $\Phi^{N-1,0}(\mathcal{E})$ is defined. Suppose there exists a morphism $\mu:C_{N-1} \longrightarrow H^*(X,Y)$ such that $j^*\circ\mu \in \Phi^{N-1,0}(\mathcal{E})$ where $j^*:H^*(X,Y) \longrightarrow H^*(X)$ is induced by inclusion. Then $\Phi^{N,0}(i^*\mathcal{E})$ is defined and for such μ there exists a class $\zeta \in \Phi^{N,0}(i^*\mathcal{E})$ such that $\delta\zeta = \Phi^{N,N-1}(\mu)$ where δ is the coboundary operator and $i:Y \subset X$ is the inclusion.

5.1.2. REMARK. There is a corresponding set of axioms for the dual complex as anti-stable Steenrod $\mathcal{A}'$- free complex where the complex is considered as right $\mathcal{A}'$-free complex. The anti-stable operation is given by

$$x(\theta) = (-1)^{\deg(\theta).\deg(x)}\,\theta(x) .$$

($\mathcal{A}'$ for mod p Steenrod algebra can be thought of as the opposite ring of $\mathcal{A}$.)

5.1.3. Definition (Spanier-Whitehead Duality).

Suppose X and Y are CW-complex. They are said to be N-dual to each other if there is a map $\varphi : X\wedge Y \longrightarrow S^N$ from the smashed product $X\wedge Y$ to the N-sphere such that

$$\varphi^*(\iota_N)/ : \tilde{H}_q(X) \longrightarrow \tilde{H}^{N-q}(Y)$$

is an isomorphism, where / is the slant homomorphism, ι_N is is the generator of $H^N(S^N)$ and H is a reduced mod p cohomology theory. The following is due to Spanier and J.H.C. Whitehead. In the suspension category of Whitehead, if X is a subcomplex of S^{N+1} and X^* is a S-deformation retract of S^{N+*}—X, then X and X^* are said to be N S-dual to each other. So we denote X^* to be the S-dual of X. Hence $X \wedge X^*$ is self dual in the S-theory. Using Alexander duality between homology and cohomology one sees that the existence of φ is equivalent to the existence of a map $\psi : S^N \longrightarrow X \wedge Y$ such that

$$\psi_*(\iota_N)/ \ : H^q(X) \longrightarrow H_{N-q}(Y)$$

is an isomorphism for all q. This is equivalent to the first definition.

Thus there is a non-singular pairing

$$\tilde{H}^r(X) \otimes \tilde{H}^{N-r}(Y) \longrightarrow H^N(S^N) \cong Z_p \ .$$

Denote the pairing by $< \ , \ >$. Then if $\theta \in \mathcal{O}_q$ define $\chi(\theta)$ as follows:

$$< \chi(\theta)x, \ y > = < x, \theta \ y >$$

for all $x \in H^r(x)$, $y \in H^{N-r-q}(X^*)$.

We shall restrict ourselves to the case p = 2. Therefore from now on H will always denote mod 2 cohomology .

5.1.4. LEMMA. The map $\chi : \mathcal{O}(2) \longrightarrow \mathcal{O}(2)$ is an anti-automorphism satisfying $\chi(Sq)Sq = 1$ where

$$Sq = \sum_{i=0}^{\infty} Sq^i \quad \text{and} \quad \chi(Sq) = \sum_{i=0}^{\infty} \chi(Sq^i)$$

are formal sums. Hence it coincides with Milnor's canonical anti-automorphism $\chi : \mathcal{A}(2) \longrightarrow \mathcal{A}(2)$

Proof. Since $< \chi(\theta_1 \theta_2)x, y > = < x, \theta_1\theta_2 y >$

$$= < \chi(\theta_1)x, \theta_2 y > = < \chi(\theta_2)\chi(\theta_1)x, y >$$

χ is an anti-homorphism. It is therefore an anti-automorphism by duality. To see that the relation is satisfied we note the following:

If $x \otimes y \in H^{N-i}(X \wedge X^*)$ then

$$0 = \psi^*(Sq^i(x \otimes y)) = \sum_{j=0}^{i} < Sq^j x, Sq^{i-j} y >$$

$$= \sum_{j=0}^{i} < \chi(Sq^{i-j}) Sq^j x, y > .$$

This is true for any i, any class $x \otimes y \in H^*(X \wedge X^*)$ and arbitrary CW-complex X. The last assertion comes from the uniqueness of stable operation.

5.1.5. Dual Chain Complex.

Suppose M is a left $\mathcal{A}(2)$-module. Define $\overline{M} = \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})$ with the right $\mathcal{A}$'-module structure given by

$(f^a)(m) = f(m)\chi(a)$ for all $f \in \overline{M}$ and for all $m \in M$.

Denote $\mathcal{A}$ to be the mod 2 Steenrod algebra for simplicity of notation. Note that f^a is a left $\mathcal{A}$-module homomorphism .

$((b \cdot f^a)(m) = bf(m)\chi(a) = f(bm)\chi(a) = f^a(bm)$. Observe that $f^{a \circ b}(m) = (f^{ba})(m) = f(m)\chi(ba) = f(m). \chi(a)\chi(b).$)

Suppose $\varphi : M \longrightarrow N$ is a map between left $\mathfrak{A}$-modules. Define

$\varphi^* : \overline{N} \longrightarrow \overline{M}$ to be the dual homomorphism in the ordinary sense by $\varphi^*(f)(m) = f(\varphi(m))$ for all $f \in \overline{N}$ and $m \in M$. This is also a right $\mathfrak{A}'$-module homomorphism since

$$\varphi^*(f^a)(m) = f^a(\varphi(m)) = (f \circ \varphi(m)).a = \varphi^*(f)(m).a$$
$$= (\varphi^*(f))^a(m) \quad \text{for all } a \in \mathfrak{A}',\ f \in \overline{N} \text{ and}$$
$$\text{all} \quad m \in M .$$

But this is <u>not</u> the homomorphism that interests us. The homomorphism that we want should be 'compatible' with 'S-duality'.

Suppose M is a finitely generated free $\mathfrak{A}$-module with generators $\{ m_i \}$. Then we have the dual basis $\{ \overline{m}_i \}$ generating the free $\mathfrak{A}'$-module $\overline{M}$. (m_i satisfies the following $\overline{m}_i(m_j) = \delta_{ij}$ where δ_{ij} is the Kronecker index homomorphism.) Let N be another finitely generated free $\mathfrak{A}$-module with generators $\{ n_i \}$. Suppose a homomorphism $\varphi : M \longrightarrow N$ is given by

$$\varphi(m_i) = \Sigma_j \, a_{ij} n_j \quad .$$

Then we define $\overline{\varphi} : \overline{N} \longrightarrow \overline{M}$ (distinct from φ^*) by

$$\overline{\varphi}(\overline{n}_j) = \Sigma_i \, \chi(a_{ij}) \, \overline{m}_i \quad .$$

Thus given a locally finitely generated free left $\mathfrak{A}$-complex

$$C: \quad C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \qquad \longleftarrow C_{N-1} \xleftarrow{d_N} C_N$$

we associate a locally finitely generated free right $\mathfrak{A}'$-complex

$$\overline{C}: \quad \overline{C}_N \xleftarrow{\overline{d}_N} \overline{C}_{N-1} \text{---} \cdots\cdots \quad \cdots \leftarrow \overline{C}_1 \xrightarrow{\overline{d}_1} \overline{C}_0 .$$

We think of this as a free finitely generated left $\mathfrak{A}$-module to avoid the unfamiliar opposite ring structure of $\mathfrak{A}$.

Now $\bar{\varphi}(\bar{m}^{a\cdot b}) = \bar{\varphi}(\bar{m}^{a})b = \bar{\varphi}(\bar{m}^{b.a})$ for all $a,b \in \mathcal{O}'$ $,\bar{m} \in \bar{N}$.
When M and N are both regarded as left $\mathcal{O}$-module $\bar{\varphi}$ is given by

$$ba\,\bar{\varphi}(\bar{m}) = \bar{\varphi}(\bar{m}^{a\cdot b}) = \varphi(ba.\bar{m}) = b\,\bar{\varphi}(a.\bar{m}).$$

It is easily seen that $\bar{C}$ is an $\mathcal{O}'$-free complex (of order two) and hence by the above consideration can be regarded as an $\mathcal{O}$-free complex. So we can associate a pyramid of cohomology operations as in 5.1.1 and is called the dual pyramid of $\Phi^{r,s}$. It is denoted by $\chi\Phi^{r,s}$.

REMARK. We should have taken into account the change of degree under the S-duality operator and hence an augmentation of degree by some integer depending on the space operated on in practice. It is usually the integer N that determine the N-dual space we want to use. It should be pointed out that the dual operations does not depend on the choice of S-dual (see C.R.F.Maunder [21]). For simplicity of notation we have denoted the dual complex without the augmentation of degree.

5.1.6. Cohomology operations and their duals.

Let $\Phi^{r,s}$ be a pyramid of cohomology operations associated with a chain complex $\mathcal{C}$ and let $\chi(\Phi^{r,s})$ be its dual. Then inductively we have the following pairings:

$\mathrm{Ker}(\Phi^{r-1,s}, X)\otimes H^*(X^*)/\mathrm{Im}(\chi(\Phi^{r-1,s}),X^*) \longrightarrow Z_2$

and $H^*(X)/\mathrm{Im}(\Phi^{r,s+1},X)\otimes\mathrm{Ker}(\chi(\Phi^{r,s+1}),X^*)$ ———

where X^* is a S-dual of X and

$$\chi(\Phi^{r,s}): \mathrm{Ker}(\chi(\Phi^{r,s+1}), X^*) \longrightarrow H^*(X^*)/\mathrm{Im}(\chi(\Phi^{r+1,s}), X^*)$$

is defined inductively by

$$< x, \chi(\Phi^{r,s})y > \; = \; < \Phi^{r,s}x, y >$$

for fixed $y \in \mathrm{Ker}(\chi(\Phi^{r,s+1}), X^*)$ and for all $x \in \mathrm{Ker}(\Phi^{r-1,s}, X)$ where $< \; , \; >$ is the inductively defined pairing and $\Phi^{r,s}$ is the pyramid associated with the portion

$$C_r \longrightarrow C_{r-1} \quad \cdots \quad \cdots \quad C_{s+1} \longrightarrow C_s \; .$$

Thinking the pairing as Alexander-Pontrjagin duality $H_n(X) \longrightarrow H^{N-n}(X^*)$ (for a sufficiently large N) and of $\chi(\theta)$ ($\theta \in \mathcal{C}_q$) being given by the vector space dual of $\bar{\theta}$ where $\bar{\theta}$ is the natural homomorphism (determined by Alexander-Pontrjagin duality) making the following diagram commutative

$$\begin{array}{ccc} H^{N-n}(X^*) & \xrightarrow{\theta} & H^{N-n+q}(X^*) \\ \approx \uparrow & & \uparrow \approx \\ H_n(X) & \xrightarrow{\bar{\theta}} & H_{n-q}(X) \end{array}$$

Alexander-Pontrjagin duality

and recalling the construction of the dual complex $\bar{\mathcal{C}}$, we see that the pyramid of cohomology operations defined above is the same as the operations associated with $\bar{\mathcal{C}}$. Hence

THEOREM (C.R.F. Maunder). $\{ \chi(\Phi)^{r,s} \}$ defines a pyramid of (stable) cohomology operations associated with the chain complex $\bar{\mathcal{C}}$ and $\chi(\Phi)^{r,s}$ is non-zero in X^* if and only if $\Phi^{r,s}$ is non-zero in X.

The proof follows from the pairings given above . For details see Maunder [21] .

where X^* is a S-dual of X and

$$\chi(\Phi^{r,s}):\mathrm{Ker}(\chi(\Phi^{r,s+1}),X^*) \to H^*(X^*)/\mathrm{Im}(\chi(\Phi^{r+1,s}),X^*)$$

is defined inductively by

$$< x, \chi(\Phi^{r,s})y > \; = \; < \Phi^{r,s}x, y >$$

for fixed $y \in \mathrm{Ker}(\chi(\Phi^{r,s+1}),X^*)$ and for all $x \in \mathrm{Ker}(\Phi^{r-1,s},X)$ where $< \; , \; >$ is the inductively defined pairing and $\Phi^{r,s}$ is the pyramid associated with the portion

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The proof follows from the pairings given above . For details see Maunder [21] .

§5.2. The Main Theorem

In this section we shall describe the background for the modification of the operation $\circledast^*$ so as to give some control over the indeterminacy of the new operation defined on mod 2 classes. We state the main theorem of this chapter.

THEOREM. Let M be a closed, connected smooth manifold of dimension $n \equiv 7\ (8) \geq 15$. Suppose M is 5-connected mod 2 $\delta w_{n-7}(M) = 0$ and $P_1(M) = 0$. Then M admits at least 7 independent tangent vector fields.

5.2.1. The Indeterminacy $\mathrm{Indet}^{2n}(\circledast^*, TM)$.

PROPOSITION. $\mathrm{Indet}^{2n}(\circledast^*, TM) = 0$ where $\circledast^*$ is the tertiary cohomology operation as in §4.4. and TM is the Thom complex of the tangent bundle of M.

Proof. By 4.2.6

$$\begin{aligned}\mathrm{Indet}^{2n}(\circledast^*, TM) &\subset \{\Gamma_2(a) \mid a \in H^{n+1}(TM)\} + \\ &\quad \{\Gamma_3(a) \mid a \in H^{2n-7}(TM)\} \quad . \\ &= \{\Gamma_3(a) \mid a \in H^{2n-7}(TM)\}\end{aligned}$$

since $\Gamma_2(a) = 0$ modulo zero indeterminacy by connectivity condition on M and Poincare duality. Γ_2 and Γ_3 are the secondary cohomology operations defined by the following relations in $\mathcal{A}$:

$$\chi(Sq^4)Sq^{n-4} + Sq^{n-2}Sq^2 = 0 \quad \text{and} \tag{5.2.2}$$

$$(Sq^2Sq^4)Sq^2 + \chi(Sq^4)Sq^4 = 0 \ . \tag{5.2.3}$$

Now the Thom space of a normal bundle ν of $M \times M$ in R^N for some sufficiently large integer N is the S-dual of $M \times M \cup$ a point. Observe that $P_1(\nu) = 0$. Thus the normal bundle ν of $M \times M$ is classified by a map into $BSpin_{N-2n}[8]$. Since $H^7(BSpin_{N-2n}[8]) \cong 0$ $\chi(\Gamma_3)$ is zero on the Thom class of the universal bundle over $BSpin_{N-2n}[8]$. By naturality $\chi(\Gamma_3)$ is zero on the Thom class of ν. Therefore by Theorem 5.1.6 Γ_3 is zero on $H^{2n-7}(M \times M)$ modulo indeterminacy and is therefore zero modulo zero indeterminacy by Wu duality.
Let $g: M \times M \longrightarrow TM$ be the map that collapses the complement of a tubular neighbourhood of the diagonal in $M \times M$ to a point. Then by the above argument $g^*(\Gamma_3(a)) = \Gamma_3(g^*(a)) = 0$ for all $a \in H^{2n-7}(TM)$ and since the indeterminacy $\mathrm{Indet}^{2n}(\Gamma_3, TM) \cong 0$ by Wu duality we conclude that $\Gamma_3(a)$ is zero for all $a \in H^{2n-7}(TM)$. Thus the proposition follows.

REMARK. $\mathrm{Indet}^{2n}(\Theta, TM)$ can be seen to be zero modulo some undetermined primary operations on U_M the Thom class of the tangent bundle of M. By Proposition 5.2.1, 4.2.6

$\mathrm{Indet}^{2n}(\Theta, TM) \subset \Gamma_1(U_M)$ modulo some primary cohomology operation where Γ_1 is associated with the relation in $\mathcal{O}\!\mathcal{L}$

$$(5.2.4) \quad (Sq^2 Sq^4) Sq^{n-5} + \chi(Sq^4) Sq^{n-3} + Sq^{n-2} Sq^3 = 0 .$$

By Theorem 4.4.3 TM is a sixth suspension and by an argument similar to Theorem 3.4.5 we see that

$$\Gamma_1(U_M) = \{ U_M \cup (w_{n-6} \cdot w_6) \} = \{ 0 \}$$
$$= 0 \text{ modulo zero indeterminacy.}$$

5.2.5. A decomposition of relation (4.1.8)

Because of Remark 5.2.1 we modify the operation Ⓗ so that it is defined on some cohomology class in $H^n(M\times M)$ by a decomposition of the relation (4.1.8). For convenience we write down the relations involved instead of the chain complex they define. We use the same symbol for the operation as well as for the relation that determines it.

$$
\begin{aligned}
\zeta_1 &: \quad Sq^2Sq^{n-6} + Sq^2(Sq^{n-7}Sq^1) = 0 \ ; \\
\zeta_2 &: \quad Sq^2Sq^{n-5} + Sq^1(Sq^{n-5}Sq^1) = 0 \ ; \\
\zeta_3 &: \quad Sq^4Sq^{n-6} + \chi(Sq^4)(Sq^{n-7}Sq^1) + \chi(Sq^4)(Sq^{n-9}Sq^2Sq^1) + \\
&\qquad + \chi(Sq^4)(Sq^{n-8}Sq^2) = 0 \ ; \\
\zeta_4 &: \quad (Sq^2Sq^1)Sq^{n-5} + Sq^1Sq^{n-3} + Sq^4(Sq^{n-7}Sq^1) + Sq^2(Sq^{n-7}Sq^2Sq^1) \\
&\qquad + Sq^1(Sq^{n-7}Sq^3Sq^1) = 0 \ ; \\
\zeta_5 &: \quad Sq^4Sq^{n-5} + Sq^4(Sq^{n-7}Sq^2) + Sq^1(Sq^{n-5}Sq^3) + Sq^2(Sq^{n-7}Sq^3Sq^1) \\
&\qquad = 0 \ ; \\
\zeta_6 &: \quad Sq^4Sq^{n-3} + Sq^2(Sq^{n-3}Sq^2) + Sq^1(Sq^{n-3}Sq^3) + Sq^1(Sq^{n-1}Sq^1) \\
&\qquad = 0 \ ; \\
\eta_1 &: \quad Sq^2(Sq^{n-5}Sq^2Sq^1) + (Sq^4Sq^2)(Sq^{n-9}Sq^2Sq^1) + \\
&\qquad + (Sq^3Sq^1)(Sq^{n-7}Sq^2Sq^1) + (Sq^4Sq^2)(Sq^{n-8}Sq^2) \\
&\qquad + Sq^2(Sq^{n-7}Sq^2Sq^3) = 0 \ ; \\
\eta_2 &: \quad Sq^3(Sq^{n-7}Sq^2Sq^1) + Sq^1(Sq^{n-5}Sq^2Sq^1) = 0 \ ; \\
\eta_3 &: \quad (Sq^2Sq^1)(Sq^{n-9}Sq^2Sq^3) + Sq^1(Sq^{n-7}Sq^2Sq^3) + \\
&\qquad + (Sq^2Sq^1)Sq^{n-7}Sq^2Sq^1 + (Sq^4Sq^2Sq^1)Sq^{n-11}Sq^2Sq^1 = 0; \\
\eta_4 &: \quad Sq^7(Sq^{n-11}Sq^2Sq^1) = 0 \ ;
\end{aligned}
$$

$$\eta_5: \quad (Sq^2Sq^1)(Sq^{n-7}Sq^1) = 0 ;$$

$$\eta_6: \quad (Sq^2Sq^1)(Sq^{n-8}Sq^2) = 0 \text{ and}$$

$$\eta_7: \quad (Sq^2Sq^1)(Sq^{n-9}Sq^2Sq^1)+(Sq^2Sq^3)(Sq^{n-11}Sq^2Sq^1) = 0 .$$

It is not hard to see that there is a relation among the above relations in $\mathfrak{A}$:

$$\Omega : \quad (Sq^2Sq^4)\zeta_1+\chi(Sq^4)\zeta_3+Sq^2\eta_1+Sq^3\eta_2+Sq^3\eta_3+Sq^3\eta_4 + Sq^5\eta_5 + Sq^5\eta_6+ Sq^5\eta_7 = 0 .$$

Using a similar argument to the proof of Theorem 3.5.1 we obtain the following:

5.2.6. LEMMA. Notation as given by Diagram (2.1.4). There exist stable secondary cohomology operations ζ_i $i = 1,\ldots,6$ such that

$$\zeta_i((Tq_1)^*(U_{BSpin_n})) = \{(Tq_1)^*(U_{BSpin_n}).k_i^2\}$$

for $i = 1,2,4,5,6$ and

$$\zeta_3((Tq_1)^*(U_{BSpin_n})) = \{(Tq_1)^*(U_{BSpin_n}).(k_3^2 + w_4.w_{n-7})\}.$$

Moreover the conclusion also holds when ζ_1 and ζ_3 is replaced by $\tilde{\zeta}_1$ and $\tilde{\zeta}_3$ which are associated with the relations $\tilde{\zeta}_1$ and $\tilde{\zeta}_3$ where Sq^{n-6} is replaced by δSq^{n-7}.

The following is easily proven:

5.2.7. LEMMA. There exists operations η_i $i = 1,\ldots,7$ such that $\eta_i(U_{BSpin_n[8]}) = \{ 0 \}$ modulo indeterminacy for $i = 1, 2,\ldots,6$ and 7.

5.2.8. Now we take the universal example for the secondary cohomology operations of 5.2.5 collectively. Furthermore we restrict the domain of the vector operation by intersecting it with the kernels of $Sq^{n-7}Sq^4Sq^1$, $Sq^{n-9}Sq^4Sq^1$, $Sq^{n-7}Sq^4Sq^2$, and $Sq^{n-8}Sq^4Sq^2$. Denote the resulting universal example by $p_{1,k}:P_{1,k} \longrightarrow K_k$. Then we see that

$$H^{k+n}(K_k) \cap \mathrm{Ker}(\sigma^{15}) \cap \{Sq^1 \iota_k, Sq^2 \iota_k\}$$
$$\subset \mathrm{Ker}(p^*_{1,k}) \cap H^{k+n}(K_k) \text{ for } k \geqslant 15.$$

5.2.9. PROPOSITION. Let $p_{1,k}:P_{1,k} \longrightarrow K_k$ be the universal example for the vector secondary cohomology operation of 5.2.5 as given by 5.2.8. Then with the choice of operations η_i for i = 1, ..., 7 satisfying the conclusion of Lemma 5.2.7 and ζ_i i= 1, 3 satisfying the conclusion of Lemma 5.2.6 there exists the following relation among them:

$$(5.2.10) \quad (Sq^2 Sq^4)\zeta_1 + \chi(Sq^4)\zeta_3 + Sq^2\eta_1 + Sq^3\eta_2 + Sq^3\eta_3 + $$
$$+ Sq^3\eta_4 + Sq^5\eta_5 + Sq^5\eta_6 + Sq^5\eta_7 = 0 .$$

Hence there is defined a family of tertiary cohomology operations Ω associated with the relation (5.2.10).

Proof. The relation Ω is essentially a decomposition of the relation giving (4.1.8). Therefore by naturality and Lemma 4.1.9 the expression on the left hand side of (5.2.10) is equal to $p^*_{1,k}(a)$ for some class $a \in H^{n+k}(K_k)$ such that $a \in \{Sq^1\iota_k, Sq^2\iota_k\}$. But by inspection we can require the representatives of the operations involved in the left hand side to satisfy that σ^{15} maps them to zero in addition to the conclusion of Lemma 5.2.7 and Lemma 5.2.6 for $k > 15$. Therefore by 5.2.8 $p^*_{1,k}(a) = 0$ for $k > 15$. It is clearly also zero when $k \leq 15$. This completes the proof of Proposition 5.2.9.

5.2.11. PROPOSITION. There exists a stable tertiary cohomology operation Ω associated with the relation (5.2.10) such that $\Omega((T\pi)^*(U_{BSpin_n}[8])) = \{0\}$ modulo indeterminacy $\mathrm{Indet}^{2n}(\Omega, TBSpin_{n-7}[8])$.

Proof. This follows from Proposition 5.2.9 and an argument similar to the proof of Theorem 4.2.9.

The following is immediate from Proposition 5.2.11:

5.2.12. COROLLARY. Let $\tilde{\Omega}$ be the operation associated with the relation (5.2.10) where we replace ζ_1 and ζ_3 by $\tilde{\zeta}_1$ and $\tilde{\zeta}_3$. Then $\tilde{\Omega}((T\pi)^*(U_{BSpin_n}[8])) = \{0\}$ modulo indeterminacy $\mathrm{Indet}^{2n}(\tilde{\Omega}, TBSpin_{n-7}[8])$.

5.2.13. Let $\tilde{\Omega}^*$ be the tertiary cohomology operation associated with the relation $(5.2.10)^*$ where all the secondary relations are replace by those valid only on integral classes , the relations ζ_1 and ζ_3 replaced by $\tilde{\zeta}^*_1$ and $\tilde{\zeta}^*_3$ and the term $(Sq^{n-8}Sq^2)$ is replaced by $(\delta Sq^{n-9}Sq^2)$.

THEOREM. Let M be a closed and connected manifold of dimension $n \equiv 7(8) \geq 15$. Suppose M is 5-connected mod 2. Let ξ be an n-plane bundle over M satisfying $w_{n-7}(\xi) = P_1(\xi) = 0$. Then ξ has at least 7 independent cross sections if and only if $0 \in \tilde{\Omega}^*(U_\xi)$.

Proof of Theorem 5.2.13. The proof is analogous to the proof of Theorem 4.4.3 part (3). It follows from Corollary 5.2.12, the Admissible Class Theorem 2.2.14, Theorem 4.4.3, the connectivity condition on M and the fact that no new non-trivial secondary indeterminacy is introduced. We leave the details to the reader.

§5.3. The Proof of Theorem 5.2.

Throughout the remainder of this chapter $g: M\times M \longrightarrow TM$ will denote the map that collapses the complement of a tubular neighbourhood of the diagonal in $M\times M$ to a point. We shall prove Theorem 5.2 as a result of a series of preliminary technical facts. We assume the hypothesis of Theorem 5.2 on M.

5.3.1. The vanishing of some Wu classes.

Suppose M' is a compact, connected and smooth q-dimensional manifold. Let $v_i \in H^i(M')$ denote the i^{th} Wu class of M'(i.e. for each class $u \in H^{n-i}(M')$ $Sq^i u = u.v_i$.). Then for dimensional reason $v_i = 0$ for $i > q/2$. Furthermore we have the following mild generalization of a result of Massey due to E. Thomas:

PROPOSITION (E. Thomas). Suppose M' is a connected , compact and smooth manifold satisfying

$w_1(M') = w_2(M') = \dots = w_{2^r}(M') = 0$ for some non-negative integer r. Then $v_i = 0$ unless $i \equiv 0 \mod 2^{r+1}$.

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Proof. The proof follows from an observation of E. Thomas namely, that there is a decomposition

$$(5.3.2)\qquad Sq^{2^{j+1}.k+2^j} = Sq^{2^j}Sq^{2^{j+1}.k} + \sum_{i=0}^{j-1} Sq^{2^i}\alpha_i$$

where degree of $\alpha_i = 2^{i+1}.m_i+2^i$ for some positive integer m_i $(0 \leq i \leq j-1)$,

for each positive integer j and for $k \geq 1$. This decomposition is readily derived from the Adem relations by induction and the fact that for each positive integer $m < 2^j$

$2^{j+1}.k-m = 2^{p+1}.l + 2^p$ where 2^p is the highest power of 2 that divides m, for some integer $l \geq 1$.

5.3.3. Let M' be an odd dimensional connected and closed manifold. Let $U_{M'}$ be the Thom class of its tangent bundle and $\underline{U} = g^*(U_{M'})$. Milnor [23] gave the following decomposition:

$$\underline{U} = \sum_{i=1}^{n/2} \alpha_i^k \otimes \beta_{n-i}^j + \sum_{i=1}^{n/2} \alpha_{n-i}^j \otimes \beta_i^k$$

where $\alpha_i^k \cup \beta_{n-i}^j = \delta_{kj}\mu$ and μ is the fundamental class of M. Let $\tau : H^*(M' \times M') \longrightarrow H^*(M' \times M')$ be the isomorphism induced by the map that interchanges the factors. Define $A \in H^n(M' \times M')$ by

$$A = \sum_{i=1}^{n/2} \alpha_i^k \otimes \beta_{n-i}^j .$$

Then $\underline{U} = A + \tau A$.

The remainder of the section is devoted to showing that $\Omega(\underline{U}) = 0$ modulo zero indeterminacy.

5.3.34. PROPOSITION. $\mathrm{Indet}^{2n}(\Omega, TM) = \mathrm{Indet}^{2n}(\Omega, M \times M)$

$= 0$.

Proof. $g^* \mathrm{Indet}^{2n}(\Omega, TM) \subset \mathrm{Indet}^{2n}(\Omega, M\times M)$. Since g^* is injective (see [6]), it is enough to show that $\mathrm{Indet}^{2n}(\Omega, M\times M) = 0$.

From 5.2.5 we see that

$$
\begin{aligned}
\mathrm{Indet}^{2n}(\Omega, M\times M) \subset & \{ \Gamma_1(a) \mid a \in H^{2n-7}(M\times M)\} + \\
& + \{ \Gamma_2(a) \mid a \in H^{2n-7}(M\times M)\} + \\
& + \{ \Gamma_3(a) \mid a \in H^{2n-7}(M\times M)\} + \\
& + \{ \Gamma_3(a) \mid a \in H^{2n-7}(M\times M)\} + \\
& + \{ \Gamma_4(a) \mid a \in H^{2n-9}(M\times M)\}
\end{aligned}
\tag{5.3.5}
$$

modulo some undetermined primary operations of degree 7 and degree 9,

where $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are defined by the following relations in $\mathfrak{A}$:

Γ_1: $(Sq^2Sq^4)Sq^2 + \chi(Sq^4)Sq^4 = 0$,

Γ_2: $(Sq^2Sq^4)Sq^2 + \chi(Sq^4)\chi(Sq^4) + Sq^5(Sq^2Sq^1) = 0$,

Γ_3: $Sq^2(Sq^4Sq^2) + \chi(Sq^4)\chi(Sq^4) + Sq^5(Sq^2Sq^1) = 0$ and

Γ_4: $Sq^3(Sq^4Sq^2Sq^1) + Sq^3Sq^7 + Sq^5(Sq^2Sq^3) = 0$.

By Lemma 5.3.1, Wu duality all the operations of degree 7 and degree 9 mapping into the top dimensional cohomology group $H^{2n}(M\times M)$ are zero. Therefore we are left with showing that Γ_i is trivial for $i = 1,2,3,4$ in the right hand side of (5.3.5). As in the proof of 5.2.1 we see that Γ_1, Γ_2 and Γ_3 are zero modulo zero indeterminacy on classes of $H^{2n-7}(M\times M)$. The same argument applies also to Γ_4 (since

$H^9(BSO_N[8]) \cong 0$ for some sufficiently large N) to conclude that Γ_4 is zero on $H^{2n-9}(M\times M)$ modulo zero indeterminacy. Thus this shows that $\mathrm{Indet}^{2n}(\Omega, M\times M) = 0$ modulo zero indeterminacy. This completes the proof of Proposition 5.3.4.

We have immediately the following:

5.3.6. PROPOSITION. $\tilde{\Omega}^*(U_M) = \Omega(U_M)$ and
$$\tilde{\Omega}^*(\underline{U}) = \Omega(\underline{U}) .$$

Furthermore $\tilde{\Omega}^*(U_M)$ is zero if and only if $\Omega(\underline{U})$ is zero.

5.3.7. We now restrict our attention to the operation Ω. Let $\approx{\Omega}$ be the tertiary cohomology operation associated with the relation (5.2.10) but where we ignore ζ_6 (that is it need not be defined on the kernel of ζ_6.

PROPOSITION. $\approx{\Omega}$ is defined on $A \in H^n(M\times M)$ as defined in 5.3.3.

Proof. We only need to show that A is in the domain of the secondary cohomology operations $\zeta_1, \zeta_2, \ldots, \zeta_4, \zeta_5, \eta_1, \eta_2, \eta_3, \ldots, \eta_6$ and η_7. This follows from the connectivity condition on M and the following proposition:

5.3.8. PROPOSITION. $Sq^{n-6}A = Sq^{n-7}Sq^1A = Sq^{n-9}Sq^2Sq^1A = 0$ and $Sq^{n-11}Sq^2Sq^1A = 0$.

Proof. $Sq^{n-6}A = 0$ follows from the fact that $Sq^{n-6}H^n(M\times M) = 0$ which is seen by using the Cartan formula, Lemma 5.3.1 and the connectivity condition on M. We shall prove only

$Sq^{n-11}Sq^2Sq^1A = 0$ the other cases are similar.

Now dim $M = n = 8s+7$ for some integer $s \geqslant 1$. By the Cartan formula, Lemma 5.3.1 and the connectivity condition on M

$$0 = Sq^{n-11}Sq^2Sq^1(A + \tau A) = Sq^{n-11}(Sq^2Sq^1A + Sq^2Sq^1\tau A)$$

$$= \Sigma\ \underbrace{Sq^{4s}a_{4s+7} \otimes Sq^{4s-1}}_{\substack{\text{bidegree}\\(8s+7,8s-1)}} b_{4s} + \Sigma\ \underbrace{Sq^{4s-1}b_{4s} \otimes Sq^{4s}}_{\substack{\text{bidegree}\\(8s-1,8s+7)}} a_{4s+7}$$

for some $a_{4s+7} \in H^{4s+7}(M)$ and $b_{4s} \in H^{4s}(M)$.

This shows that $Sq^{n-11}Sq^2Sq^1A = 0 = Sq^{n-11}Sq^2Sq^1\tau A$.

Similarly $Sq^{n-7}Sq^1A = Sq^{n-9}Sq^2Sq^1A = Sq^{n-8}Sq^2A = 0$.

5.3.9. Proof of Theorem 5.2.

By Proposition 5.3.4 $\mathrm{Indet}^{2n}(\widetilde{\widetilde{\Omega}}, M\times M) = 0$ modulo zero indeterminacy. Therefore $\Omega(\underline{U}) = \widetilde{\widetilde{\Omega}}(\underline{U})$ modulo zero indeterminacy. Thus

$$\widetilde{\widetilde{\Omega}}(\underline{U}) = \widetilde{\widetilde{\Omega}}(A + \tau A) = \widetilde{\widetilde{\Omega}}(A) + \widetilde{\widetilde{\Omega}}(\tau A)$$

$$= 0 \text{ modulo zero indeterminacy.}$$

Therefore this implies that $\Omega(\underline{U}) = 0$. Hence $\breve{\Omega}^*(U_M) = \Omega(U_M) = 0$. Thus by Theorem 5.2.13, M has at least 7 independent tangent vector fields. This completes the proof of Theorem 5.2.

REMARK. $\tilde{\Omega}^*(U_M) = 0$ implies that $0 \in \textcircled{H}^*(U_M)$ follows from the proof of this theorem. Alternatively one observes

$Sq^{n-11}Sq^2Sq^1A = 0$ the other cases are similar.

Now dim $M = n = 8s+7$ for some integer $s \geqslant 1$. By the Cartan formula, Lemma 5.3.1 and the connectivity condition on M

$$0 = Sq^{n-11}Sq^2Sq^1(A + \tau A) = Sq^{n-11}(Sq^2Sq^1A + Sq^2Sq^1\tau A)$$

$$= \Sigma \underbrace{Sq^{4s}a_{4s+7} \otimes Sq^{4s-1}b_{4s}}_{\substack{\text{bidegree} \\ (8s+7,8s-1)}} + \Sigma \underbrace{Sq^{4s-1}b_{4s} \otimes Sq^{4s}a_{4s+7}}_{\substack{\text{bidegree} \\ (8s-1,8s+7)}}$$

for some $a_{4s+7} \in H^{4s+7}(M)$ and $b_{4s} \in H^{4s}(M)$.

This shows that $Sq^{n-11}Sq^2Sq^1A = 0 = Sq^{n-11}Sq^2Sq^1\tau A$.

Similarly $Sq^{n-7}Sq^1A = Sq^{n-9}Sq^2Sq^1A = Sq^{n-8}Sq^2A = 0$.

5.3.9. Proof of Theorem 5.2.

By Proposition 5.3.4 $\mathrm{Indet}^{2n}(\tilde{\tilde{\Omega}}, M\times M) = 0$ modulo zero indeterminacy. Therefore $\Omega(\underline{U}) = \tilde{\tilde{\Omega}}(\underline{U})$ modulo zero indeterminacy. Thus

$$\tilde{\tilde{\Omega}}(\underline{U}) = \tilde{\tilde{\Omega}}(A + \tau A) = \tilde{\tilde{\Omega}}(A) + \tilde{\tilde{\Omega}}(\tau A)$$

$$= 0 \text{ modulo zero indeterminacy.}$$

Therefore this implies that $\Omega(\underline{U}) = 0$. Hence $\tilde{\Omega}^*(U_M) = \Omega(U_M) = 0$. Thus by Theorem 5.2.13, M has at least 7 independent tangent vector fields. This completes the proof of Theorem 5.2.

REMARK. $\tilde{\Omega}^*(U_M) = 0$ implies that $0 \in \Theta^*(U_M)$ follows from the proof of this theorem. Alternatively one observes that the representative of the operation $\sigma^{6}\Theta^{*}$ is actually determined by a representative for $\sigma^{6}\tilde{\Omega}^{*}$. By Theorem 4.4.3 TM is a sixth suspension. This and the proof of Theorem 5.2 shows that $0 \in \Theta^{*}(U_{M})$.

§5.4. A Cohomology Operation related to $\mathcal{C}_n$.

In this section we make an observation about the vanishing of a tertiary cohomology operation related to Θ of Chapter 4 on the Thom class of the universal j-plane bundle on $BSO_j[8]$ $j \geqslant 8$. This will help to understand the motivation behind the thinking of the last three sections.

5.4.1. Definition. Define the chain complex of free $\mathcal{A}$-modules and $\mathcal{A}$-homomorphisms

$$\tilde{\mathcal{C}}_n : \quad \tilde{C}_0 \xrightarrow{d_1} \tilde{C}_1 \xrightarrow{d_2} \tilde{C}_2 \xrightarrow{d_3} \tilde{C}_3$$

by

$\tilde{C}_0$ is free on one generator c_0 of degree 0;

$\tilde{C}_1$ is free on generators $\{ c_{1,1}, c_{1,2}, c_{1,n-6} \}$ where degree of $c_{1,i} = i$;

$\tilde{C}_2$ is free on generators $\{ c_{2,n}, c_{2,n-2}, c_{2,n+1} \}$ where degree of $c_{2,i} = i$;

$\tilde{C}_3$ is free on one generator $c_{3,n+2}$ of degree $n+2$;

and the differentials are defined below:

$$d_1 c_{1,1} = Sq^1 c_0 \ ; \ d_1 c_{1,2} = Sq^2 c_0; \ d_1 c_{1,n-6} = Sq^{n-6} c_0;$$

$$d_2 c_{2,n} = Sq^4 Sq^2 c_{1,n-6} + Sq^4 Sq^{n-5} c_{1,1};$$

$$d_2 c_{2,n-2} = Sq^4 c_{1,n-6} + Sq^{n-3} c_{1,1} + Sq^{n-4} c_{1,2};$$

$$d_2 c_{2,n+1} = Sq^{n-3} Sq^3 c_{1,1} + Sq^{n-3} Sq^2 c_{1,2} \ ; \text{ and}$$

$$d_3 c_{3,n+2} = Sq^2 c_{2,n} + (Sq^4) c_{2,n-2} + Sq^1 c_{2,n+1} .$$

Thus we see that $\tilde{\mathcal{C}}_n$ is a modification of $\mathcal{C}_n$.

5.4.2. PROPOSITION. $\widetilde{\mathcal{E}}_n$ is admissible. Furthermore it defines a tertiary cohomology operation $\widetilde{\Theta}$ associated with the relation

(5.4.3) $Sq^2(Sq^4Sq^2\phi_1) + \chi(Sq^4)\phi_3 + Sq^1(Sq^{n-3}\psi_1) = 0$

where ϕ_1, ϕ_2 and ψ_1 are the same as in §4.2.

Proof. This follows from Lemma 4.1.3 and lemma 4.1.9 gives the relation (5.4.3).

5.4.4. Let $\widetilde{\Theta}^*$ be the corresponding tertiary cohomology operation associated with $\widetilde{\mathcal{E}}_n^*$ obtained from $\widetilde{\mathcal{E}}_n$ by replacing C_0 by $C_0^* \cong \mathcal{O}\!\!\mathcal{L}/\mathcal{O}\!\!\mathcal{L}\, Sq^1$. Then we have the following proposition:

PROPOSITION. $\widetilde{\Theta}^*(U_M) = \Theta^*(U_M)$.

Proof. Since M is assumed to satisfy the hypothesis of Theorem 5.2, this follows from Proposition 5.2.1 and the connectivity condition on M.

5.4.5. The Dual Complex $\chi(\widetilde{\mathcal{E}}_n)$.

The dual complex of $\widetilde{\mathcal{E}}_n$ is defined by 5.1.1 as follows:

$$\chi(\widetilde{\mathcal{E}}_n): \quad \bar{\tilde{C}}_3 \xleftarrow{\bar{d}_3} \bar{\tilde{C}}_2 \xleftarrow{\bar{d}_2} \bar{\tilde{C}}_1 \xleftarrow{\bar{d}_1} \bar{\tilde{C}}_0$$

where

$$\bar{d}_3\bar{c}_{2,n} = Sq^2\bar{c}_{3,n+2}; \quad \bar{d}_3\bar{c}_{2,n-2} = Sq^4\bar{c}_{3,n+2};$$

$$\bar{d}_3\bar{c}_{2,n+1} = Sq^1\bar{c}_{3,n+2};$$

$$\bar{d}_2\bar{c}_{1,1} = \chi(S^{n-5})\chi(Sq^4)\bar{c}_{2,n} + \chi(Sq^{n-3})\bar{c}_{2,n-2} + \chi(Sq^3)\chi(Sq^{n-3})\bar{c}_{2,n+1};$$

$$\bar{d}_2\bar{c}_{1,2} = \chi(Sq^{n-4})\bar{c}_{2,n-2} + Sq^2\chi(Sq^{n-5})\bar{c}_{2,n+1};$$

$$\bar{d}_2\bar{c}_{1,n-6} = Sq^2\chi(Sq^4)\bar{c}_{2,n} + \chi(Sq^4)\bar{c}_{2,n-2} \text{ and}$$

$$\bar{d}_1\bar{c}_0 = Sq^1\bar{c}_{1,1} + Sq^2\bar{c}_{1,2} + \chi(Sq^{n-6})\bar{c}_{1,n-6} \,.$$

PROPOSITION. $\chi(\mathcal{C}_n)$ is admissible in the sense of C.R.F. Maunder.

Proof. Since $\widetilde{\mathcal{C}}_n$ is admissible this follows from Theorem 4.3.1 of Maunder [21].

This result however does not tell us anything about the choice of the operations involved. We shall choose some suitable choice for the secondary cohomology operation associated with the portion

$$\bar{\widetilde{C}}_3 \xleftarrow{\bar{d}_3} \bar{\widetilde{C}}_2 \xleftarrow{\bar{d}_2} \bar{\widetilde{C}}_1$$

of $\chi(\widetilde{\mathcal{C}}_n)$. This gives a vector secondary cohomology operation (Ψ_1, Ψ_2, Ψ_3) where Ψ_1, Ψ_2, and Ψ_3 are the secondary cohomology operations associated with the relations:

$$\Psi_1: \quad (\chi(Sq^{n-5})\chi(Sq^4))Sq^2 + \chi(Sq^{n-3})Sq^4 + (Sq^2Sq^1 + (Sq^2Sq^1\chi(Sq^{n-3}))Sq^1 = 0 \,;$$

$$\Psi_2: \quad \chi(Sq^{n-4})Sq^4 + (Sq^2\chi(Sq^{n-3}))Sq^1 = 0 \text{ and}$$

$$\Psi_3: \quad (Sq^2\chi(Sq^4))Sq^2 + \chi(Sq^4)Sq^4 = 0 \,.$$

Therefore an operation $\chi(\widetilde{\textcircled{H}})$ will be associated with a relation of the form

$$\chi(\widetilde{\textcircled{H}}): \quad Sq^1\Psi_1 + Sq^2\Psi_2 + \chi(Sq^{n-6})\Psi_3 = 0 \,.$$

From some tedious manipulation of the Adem relations in $\mathcal{A}$ we derived the following proposition:

5.4.6. PROPOSITION. The operations Ψ_i $i = 1,2$ and 3 can be chosen in such a way that

$$\Psi_1 = \chi(Sq^{n-7})Sq^6 g_1 + (Sq^2\chi(Sq^{n-5}) + Sq^1\chi(Sq^{n-7})Sq^2Sq^1$$
$$+ \chi(Sq^{n-15})Sq^4Sq^7Sq^1 + Sq^1\chi(Sq^{n-5})Sq^1$$
$$+ \chi(Sq^{n-7})Sq^3Sq^1)g_2$$
$$+ (\chi(Sq^{n-11}Sq^5Sq^2 + Sq^3\chi(Sq^{n-7}))g_3$$
$$+ (\chi(Sq^{n-7}) + \chi(Sq^{n-11})Sq^4 + \chi(Sq^{n-15})Sq^7Sq^1)g_4,$$

$$\Psi_2 = (\chi(Sq^{n-15})Sq^7Sq^3Sq^1 + Sq^3\chi(Sq^{n-7}))g_2$$
$$+ Sq^2\chi(Sq^{n-7})g_3 + \chi(Sq^{n-8})g_4 \quad \text{and}$$

$$\Psi_3 = Sq^6 g_1 + Sq^3Sq^1 g_2 + (Sq^3 + Sq^2Sq^1)g_3 + g_4$$

where g_1, g_2, g_3 and g_4 are the secondary cohomology operations defined by

g_1: $Sq^1Sq^1 = 0$,

g_2: $Sq^2Sq^2 + Sq^3Sq^1 = 0$,

g_3: $(Sq^2Sq^1)Sq^2 + Sq^4Sq^1 + Sq^1Sq^4 = 0$ and

g_4: $Sq^4Sq^4 + Sq^6Sq^2 + Sq^7Sq^1 = 0$

respectively.

The following is a well known result. I believe it is due to L. Kristensen.

5.4.7. PROPOSITION. In the stable Postnikov system $E_1 \longrightarrow K_N$ ($N \geq 15$) with k-invariant $(Sq^1\iota_N, Sq^2\iota_N, Sq^4\iota_N)$ we have the following generating set of relations (in $H^*(E_1)$):

$$R_1: \quad Sq^1 g_1 = 0 \,,$$

$$R_2: \quad Sq^4 g_1 + Sq^2 g_2 + Sq^1 g_3 = 0 \,,$$

$$R_3: \quad (Sq^7 + Sq^4 Sq^2 Sq^1) g_2 + Sq^2 Sq^1 g_4 + Sq^6 g_3 = 0 \text{ and}$$

$$R_4: \quad Sq^{13} g_1 + Sq^{11} g_2 + Sq^7 g_4 = 0.$$

Thus the $\mathcal{A}$-module structure of $H^*(E_1)$ is completely determined.

We defer the proof of Proposition 5.4.7 to the end of this section.

The next proposition gives us the choice of the operation associated with $\chi(\tilde{\mathcal{C}}_n)$ that we wanted. It is a consequence of some tedious manipulation with the Adem relations in $\mathcal{A}$.

5.4.8. PROPOSITION. Let the secondary cohomology operations Ψ_1, Ψ_2 and Ψ_3 be given by Proposition 5.4.6. Then there is a choice of representative for the tertiary cohomology operation $\chi(\tilde{\Theta})$ associated with the chain complex $\chi(\tilde{\mathcal{C}}_n)$ such that

$$\begin{aligned}\chi(\tilde{\Theta}) = {} & \chi(Sq^{n-13})\eta_4 + (Sq^3\chi(Sq^{n-7}) + Sq^1\chi(Sq^{n-5}) + \\ & + \chi(Sq^{n-15})Sq^8Sq^2Sq^1 + \chi(Sq^{n-7})Sq^2Sq^1)\eta_2 \\ & + (\chi(Sq^{n-15})Sq^8Sq^6 + \chi(Sq^{n-7})Sq^6 \\ & + \chi(Sq^{n-15})Sq^{14})\eta_1.\end{aligned}$$

whenever the right hand side is defined where η_i $i = 1, 2$ and 4 are the (stable) tertiary cohomology operations associated with the relations R_i $i = 1,2$ and 4 of Proposition 5.4.7 respectively.

5.4.9. THEOREM. $\chi(\tilde{\circledast})(U_{BSpin_N[8]}) = \{0\}$ modulo indeterminacy Indet($\chi(\tilde{\circledast})$, $TBSpin_N[8]$) where $TBSpin_N[8]$ is the Thom space of the Universal N-plane bundle over $BSpin_N[8]$ $N \geq 15$ and $U_{BSpin_N[8]}$ is its Thom class.

Proof. For dimensional reason $g_i(U_{BSpin_N[8]}) = 0$ for $i = 1,2,3$ and 4. Hence η_i $i = 1,2$ and 4 are defined on $U_{BSpin_N[8]}$. Again for dimensional reason $\eta_i(U_{BSpin_N[8]}) = 0$ for $i = 1,2$ and 4 and $N \geq 15$. The Theorem then follows from Proposition 5.4.8.

5.4.10. The Proof of Proposition 5.4.7.

The relations R_i $i = 1,2$ and 3 are easily established by using the exact sequence for the principal fibration $p: E_1 \longrightarrow K_N$ $N \geq 15$. By exactness

$$Sq^{13}g_1(\iota_N) + Sq^{11}g_2(\iota_N) + Sq^7g_4(\iota) = \alpha \cdot Sq^{14}(\iota)$$

where $\alpha \in Z_2$ and ι is the fundamental class of E_1. To see that $\alpha = 0$ we apply the above equation to $U_{BSpin_N[8]}$ to give

$$\begin{aligned} 0 &= Sq^{13}g_1(U_{BSpin_N[8]}) + Sq^{11}g_2(U_{BSpin_N[8]}) \\ &\quad + Sq^7g_4(U_{BSpin_N[8]}) \quad \text{(for dimensional reason)} \\ &= Sq^{14}(U_{BSpin_N[8]}) = U_{BSpin_N[8]} \cdot w_{14}. \end{aligned}$$

Since $w_{14} \neq 0$ we conclude that $\alpha = 0$. This shows that the relation R_4 exists. Hence this completes the proof of Proposition 5.4.7.

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